CS 4110

Programming Languages & Logics

Lecture 23
Advanced Types

Review

We’ve developed a type system for the λ-calculus and mathematical tools for proving its type soundness.

We also know how to extend the λ-calculus with new language features.

Today, we’ll extend our *type system* with features commonly found in real-world languages: products, sums, and references.

Products (Pairs)

Syntax

$$e ::= \cdots \mid (e_1, e_2) \mid \#1\, e \mid \#2\, e$$
$$v ::= \cdots \mid (v_1, v_2)$$

Products (Pairs)

Syntax

$$
\begin{aligned}
e &::= \cdots \mid (e_1, e_2) \mid \#1\, e \mid \#2\, e \\
v &::= \cdots \mid (v_1, v_2)
\end{aligned}
$$

Semantics

$$E ::= \cdots \mid (E, e) \mid (v, E) \mid \#1\, E \mid \#2\, E$$

$$\frac{}{\#1\,(v_1, v_2) \rightarrow v_1} \qquad \frac{}{\#2\,(v_1, v_2) \rightarrow v_2}$$

Product Types

$$\tau_1 \times \tau_2$$

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$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

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$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#1\, e : \tau_1}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#2\, e : \tau_2}$$

Sums (Tagged Unions)

Syntax

$$e ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ e \mid \mathsf{inr}_{\tau_1+\tau_2}\ e \mid (\mathsf{case}\ e_1\ \mathsf{of}\ e_2 \mid e_3)$$
$$v ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ v \mid \mathsf{inr}_{\tau_1+\tau_2}\ v$$

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Semantics

$$E ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ E \mid \mathsf{inr}_{\tau_1+\tau_2}\ E \mid (\mathsf{case}\ E\ \mathsf{of}\ e_2 \mid e_3)$$

$$\frac{}{\mathsf{case}\ \mathsf{inl}_{\tau_1+\tau_2}\ v\ \mathsf{of}\ e_2 \mid e_3 \rightarrow e_2\ v}$$

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Sum Types

$$\tau ::= \cdots \mid \tau_1 + \tau_2$$

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$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma \vdash e_1 : \tau_1 \to \tau \quad \Gamma \vdash e_2 : \tau_2 \to \tau}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ e_1 \mid e_2 : \tau}$$

Example

$$\begin{aligned}&\text{let } f = \lambda a : \mathbf{int} + (\mathbf{int} \to \mathbf{int}).\\ &\qquad \text{case } a \text{ of } (\lambda y : \mathbf{int}.\, y + 1) \mid (\lambda g : \mathbf{int} \to \mathbf{int}.\, g\, 35) \text{ in}\\ &\text{let } h = \lambda x : \mathbf{int}.\, x + 7 \text{ in}\\ &f\,(\mathrm{inr}_{\mathbf{int}+(\mathbf{int}\to\mathbf{int})}\; h)\end{aligned}$$

References

Syntax

$$e ::= \cdots \mid \mathsf{ref}\, e \mid !e \mid e_1 := e_2 \mid \ell$$
$$v ::= \cdots \mid \ell$$

References

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\begin{aligned}
e &::= \cdots \mid \mathsf{ref}\ e \mid {!e} \mid e_1 := e_2 \mid \ell \\
v &::= \cdots \mid \ell
\end{aligned}
$$

Semantics

$$E ::= \cdots \mid \mathsf{ref}\ E \mid {!E} \mid E := e \mid v := E$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\ v \rangle \to \langle \sigma[\ell \mapsto v], \ell \rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\ !\ell \rangle \to \langle \sigma, v \rangle}$$

$$\frac{}{\langle \sigma, \ell := v \rangle \to \langle \sigma[\ell \mapsto v], v \rangle}$$

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Question

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Store Typings

Let Σ range over partial functions from locations to types.

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$$\frac{\Sigma(\ell) = \tau}{\Gamma, \Sigma \vdash \ell : \tau\ \mathbf{ref}}$$

Reference Types Metatheory

Definition

Store σ is *well-typed* with respect to typing context Γ and store typing Σ, written $\Gamma, \Sigma \vdash \sigma$, if $dom(\sigma) = dom(\Sigma)$ and for all $\ell \in dom(\sigma)$ we have $\Gamma, \Sigma \vdash \sigma(\ell) : \Sigma(\ell)$.

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Theorem (Type soundness)

If $\cdot, \Sigma \vdash e : \tau$ and $\cdot, \Sigma \vdash \sigma$ and $\langle e, \sigma \rangle \rightarrow^* \langle e', \sigma' \rangle$ and $\langle e', \sigma' \rangle \not\rightarrow$, then e' is a value.

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If $\cdot, \Sigma \vdash e : \tau$ and $\cdot, \Sigma \vdash \sigma$ and $\langle e, \sigma \rangle \to^* \langle e', \sigma' \rangle$ and $\langle e', \sigma' \rangle \not\to$, then e' is a value.

Lemma (Preservation)

If $\Gamma, \Sigma \vdash e : \tau$ and $\Gamma, \Sigma \vdash \sigma$ and $\langle e, \sigma \rangle \to \langle e', \sigma' \rangle$ then there exists some $\Sigma' \supseteq \Sigma$ such that $\Gamma, \Sigma' \vdash e' : \tau$ and $\Gamma, \Sigma' \vdash \sigma'$.

Landin's Knot

Using references, we (re)gain the ability define recursive functions!

$$\textbf{let } r = \text{ref } \lambda x\!:\!\textbf{int}.\ 0 \textbf{ in}$$

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\textbf{let } a = (r := f) \textbf{ in} \\
f\ 5
\end{array}
$$

Fixed Points

Syntax

$$e ::= \cdots \mid \mathsf{fix}\, e$$

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$$e ::= \cdots \mid \mathsf{fix}\, e$$

Semantics

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$$\frac{}{\mathsf{fix}\ \lambda x\!:\!\tau.\, e \rightarrow e\{(\mathsf{fix}\ \lambda x\!:\!\tau.\, e)/x\}}$$

Fixed Points

Syntax

$$e ::= \cdots \mid \mathsf{fix}\; e$$

Semantics

$$E ::= \cdots \mid \mathsf{fix}\; E$$

$$\frac{}{\mathsf{fix}\; \lambda x\!:\!\tau.\, e \to e\{(\mathsf{fix}\; \lambda x\!:\!\tau.\, e)/x\}}$$

The typing rule for fix is on the homework...