

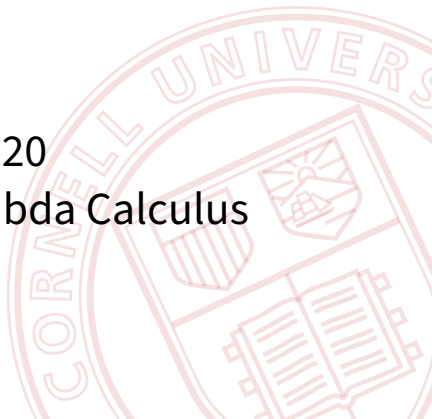
CS 4110

# Programming Languages & Logics

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Lecture 20

Simply-Typed Lambda Calculus



# Simply-Typed Lambda Calculus

## Syntax

expressions	$e ::= x \mid \lambda x:\tau. e \mid e_1 e_2 \mid n \mid e_1 + e_2 \mid ()$
values	$v ::= \lambda x:\tau. e \mid n \mid ()$
types	$\tau ::= \mathbf{int} \mid \mathbf{unit} \mid \tau_1 \rightarrow \tau_2$

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## Dynamic Semantics

$$E ::= [\cdot] \mid E e \mid v E \mid E + e \mid v + E$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x:\tau. e) v \rightarrow e\{v/x\}}$$

$$\frac{n = n_1 + n_2}{n_1 + n_2 \rightarrow n}$$

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$$\frac{\Gamma \vdash e_1 : \mathbf{int} \quad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1 + e_2 : \mathbf{int}} \text{T-ADD}$$

# Simply-Typed Lambda Calculus

## Static Semantics

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$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-VAR}$$



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$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \text{T-ABS}$$

# Simply-Typed Lambda Calculus

## Static Semantics

$$\frac{}{\Gamma \vdash n : \mathbf{int}} \text{T-INT}$$

$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{T-UNIT}$$

$$\frac{\Gamma \vdash e_1 : \mathbf{int} \quad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1 + e_2 : \mathbf{int}} \text{T-ADD}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-VAR}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \text{T-ABS}$$

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'} \text{T-APP}$$

# Properties

## Theorem (Type soundness)

*If  $\vdash e:\tau$  and  $e \rightarrow^* e'$  and  $e' \not\rightarrow$  then  $e'$  is a value and  $\vdash e':\tau$ .*

# Properties

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## Lemma (Preservation)

*If  $\vdash e:\tau$  and  $e \rightarrow e'$  then  $\vdash e':\tau$ .*

# Properties

## Theorem (Type soundness)

*If  $\vdash e : \tau$  and  $e \rightarrow^* e'$  and  $e' \not\rightarrow$  then  $e'$  is a value and  $\vdash e' : \tau$ .*

## Lemma (Preservation)

*If  $\vdash e : \tau$  and  $e \rightarrow e'$  then  $\vdash e' : \tau$ .*

## Lemma (Progress)

*If  $\vdash e : \tau$  then either  $e$  is a value or there exists an  $e'$  such that  $e \rightarrow e'$ .*

# Extra Lemmas for Preservation

## Lemma (Substitution)

*If  $x:\tau' \vdash e:\tau$  and  $\vdash v:\tau'$  then  $\vdash e\{v/x\}:\tau$ .*

## Lemma (Context)

*If  $\vdash E[e]:\tau$  and  $\vdash e:\tau'$  and  $\vdash e':\tau'$  then  $\vdash E[e']:\tau$ .*

# Extra Lemma for Progress

## Lemma (Canonical Forms)

*If  $\vdash v : \tau$ , then*

- 1. If  $\tau$  is **int**, then  $v$  is a constant, i.e., some  $c$ .*
- 2. If  $\tau$  is  $\tau_1 \rightarrow \tau_2$ , then  $v$  is an abstraction, i.e.,  $\lambda x : \tau_1. e$  for some  $x$  and  $e$ .*