CS 4110

Programming Languages & Logics

Lecture 4
Large-Step Semantics

Review

So far we've:

- Defined a simple language of arithmetic expressions
- Formalized its semantics as a "small-step" relation:
  $\langle \sigma, e \rangle \to \langle \sigma', e' \rangle$ and $\langle \sigma, e \rangle \to^* \langle \sigma', e' \rangle$

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Today we'll:

- Develop an alternate semantics based on a "large-step" relation
- Prove the equivalence of the two semantics

Large-Step Semantics

Idea: Define a new relation that captures the *complete* evaluation of an expression.

Formally: Define a relation $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$. Our new $\Downarrow$ binary relation has this type:

$$\Downarrow \subseteq (\textbf{Store} \times \textbf{Exp}) \times (\textbf{Store} \times \textbf{Int})$$

Intuition: Completely evaluating the expression e in store σ produces the number n while updating the store to σ'.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle}\ \textsc{Var}$$

Integers

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle}\ \textsc{Int}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \textsc{Add}$$

Multiplication

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \textsc{Mul}$$

Assignment

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 \,;\, e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \textsc{Assgn}$$

Large-Step Semantics

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle}\ \textsc{Int} \qquad \frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle}\ \textsc{Var}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \qquad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle}\ \textsc{Add}$$

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$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \qquad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 \,;\, e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}\ \textsc{Assgn}$$

Example

Assume that $\sigma(bar) = 7$. Let $\sigma' = \sigma[foo \mapsto 3]$.

$$\dfrac{\dfrac{}{\langle \sigma, 3\rangle \Downarrow \langle \sigma, 3\rangle}\ \textsc{Int} \qquad \dfrac{\dfrac{}{\langle \sigma', foo\rangle \Downarrow \langle \sigma', 3\rangle}\ \textsc{Var} \qquad \dfrac{}{\langle \sigma', bar\rangle \Downarrow \langle \sigma', 7\rangle}\ \textsc{Var}}{\langle \sigma', foo * bar\rangle \Downarrow \langle \sigma', 21\rangle}\ \textsc{Mul}}{\langle \sigma, foo := 3\,;\ foo * bar\rangle \Downarrow \langle \sigma', 21\rangle}\ \textsc{Assgn}$$

Equivalence

Theorem (Equivalence of small-step and large-step)

$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$ *if and only if* $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$

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Recall this definition of the multi-step relation:

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle} \textsc{Refl}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \qquad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle} \textsc{Trans}$$

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Lemma

1. *If* $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$, *then:*
   - $\langle \sigma, e + e_2 \rangle \rightarrow^* \langle \sigma', n + e_2 \rangle$
   - $\langle \sigma, n_1 + e \rangle \rightarrow^* \langle \sigma', n_1 + n \rangle$
   - $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
   - $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
   - $\langle \sigma, x := e \,;\, e_2 \rangle \rightarrow^* \langle \sigma', x := n \,;\, e_2 \rangle$

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2. *If* $\langle \sigma, e \rangle \to^* \langle \sigma', e' \rangle$ *and* $\langle \sigma', e' \rangle \to^* \langle \sigma'', e'' \rangle$*, then* $\langle \sigma, e \rangle \to^* \langle \sigma'', e'' \rangle$

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3. *If* $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$ *and* $\langle \sigma'', e'' \rangle \Downarrow \langle \sigma', n \rangle$*, then* $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$