CS 4110

Programming Languages & Logics

Lecture 2
Introduction to Semantics

Semantics

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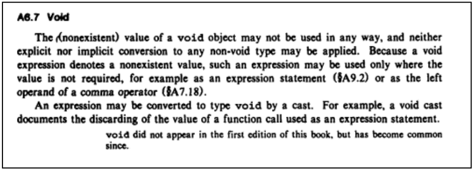
Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...


A6.7 Void

The (nonexistent) value of a void object may not be used in any way, and neither explicit nor implicit conversion to any non-void type may be applied. Because a void expression denotes a nonexistent value, such an expression may be used only where the value is not required, for example as an expression statement (§A9.2) or as the left operand of a comma operator (§A7.18).

An expression may be converted to type void by a cast. For example, a void cast documents the discarding of the value of a function call used as an expression statement.

void did not appear in the first edition of this book, but has become common since.

...but none of these is a satisfactory solution.

Formal Semantics

Three Approaches

- Operational $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters

- Denotational: $[\![e]\!]$
  - Model program as mathematical objects
  - Useful for theoretical foundations

- Axiomatic $\vdash \{\phi\}\, e\, \{\psi\}$
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness

Arithmetic Expressions

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BNF Grammar:

$$\begin{aligned} e ::= {} & x \\ \mid {} & n \\ \mid {} & e_1 + e_2 \\ \mid {} & e_1 * e_2 \\ \mid {} & x := e_1 \,;\; e_2 \end{aligned}$$

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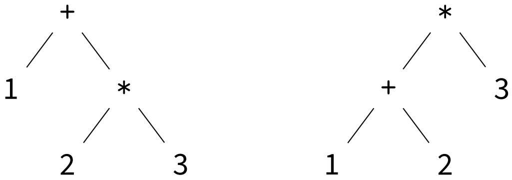

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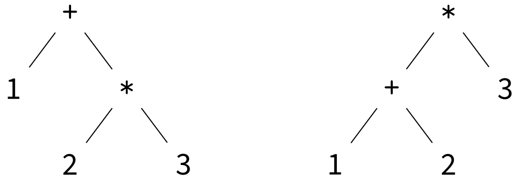

In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).

Representing Expressions

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OCaml:

```
type exp = Var of string
         | Int of int
         | Add of exp * exp
         | Mul of exp * exp
         | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var "foo", Int 1))

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Java:

```
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))

Quiz

- $7 + (4 * 2)$ evaluates to ...?

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The rest of this lecture will make these intuitions precise...

Mathematical Preliminaries

Binary Relations

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Some Important Relations

- empty: $\emptyset$
- total: $A \times B$
- identity on A: $\{(a, a) \mid a \in A\}$.
- composition $R; S$: $\{(a, c) \mid \exists b.\ (a, b) \in R \wedge (b, c) \in S\}$

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The *image* of f is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. Formally:

$$\text{image}(f) \triangleq \{f(a) \mid a \in A\}$$

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Given two functions $f : A \to B$ and $g : B \to C$, the composition of f and g is defined by: $(g \circ f)(x) = g(f(x))$ Note order!

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A function $f : A \to B$ is said to be *surjective* (or *onto*) if and only if the image of f is B.

Operational Semantics

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More formally:

$$\begin{array}{rcl} \textbf{Store} & \triangleq & \textbf{Var} \rightharpoonup \textbf{Int} \\ \textbf{Config} & \triangleq & \textbf{Store} \times \textbf{Exp} \end{array}$$

(A store is a *partial* function from variables to integers.)

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which means $(\langle \sigma, e \rangle, \langle \sigma', e' \rangle) \in$ "$\rightarrow$".

Question: How should we define this relation? Remember that there are an infinite number of configurations and possible steps!

Inference Rules

Answer: Define it inductively, using inference rules:

$$\frac{\text{premise}_1 \qquad \text{premise}_2 \qquad \cdots}{\text{conclusion}}\ \textsc{Name}$$

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An inference rule defines an implication: if all the premises hold, then the conclusion also holds.

Formally, "$\rightarrow$" is the smallest relation that is closed under all the inference rules.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle}\ \textsc{Var}$$

Addition

$$\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle}\ \textsc{Add}$$

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Multiplication

$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \to \langle \sigma, p \rangle} \textsc{Mul}$$

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Assignment

$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n\,;\ e_2 \rangle \to \langle \sigma', e_2 \rangle}\ \textsc{Assgn}$$

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$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 * e_2 \rangle \to \langle \sigma', e_1' * e_2 \rangle} \text{ LMul} \qquad \frac{\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle}{\langle \sigma, n * e_2 \rangle \to \langle \sigma', n * e_2' \rangle} \text{ RMul}$$

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$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n ;\ e_2 \rangle \to \langle \sigma', e_2 \rangle} \text{ Assgn}$$

Multi-Step Evaluation

We can define the multi-step evaluation relation, written $\rightarrow^*$, as the reflexive and transitive closure of the small-step evaluation relation.

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle} \text{ REFL}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \qquad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle} \text{ TRANS}$$