



## 1 Large-step operational semantics

In the last lecture we defined a semantics for our language of arithmetic expressions using a small-step evaluation relation  $\rightarrow \subseteq \mathbf{Config} \times \mathbf{Config}$  (and its reflexive and transitive closure  $\rightarrow^*$ ). In this lecture we will explore an alternative approach—*large-step* operational semantics—which yields the final result of evaluating an expression directly.

Defining a large-step semantics boils down to specifying a relation  $\Downarrow$  that captures the evaluation of an expression. The  $\Downarrow$  relation has the following type:

$$\Downarrow \subseteq (\mathbf{Store} \times \mathbf{Exp}) \times (\mathbf{Store} \times \mathbf{Int}).$$

We write  $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$  to indicate that  $((\sigma, e), (\sigma', n)) \in \Downarrow$ . In other words, the expression  $e$  with store  $\sigma$  evaluates in one big step to the final store  $\sigma'$  and integer  $n$ .

We define the relation  $\Downarrow$  inductively, using inference rules:

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle} \text{INT} \qquad \frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle} \text{VAR}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{ADD}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{MUL}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \text{ASSGN}$$

To illustrate the use of these rules, consider the following proof tree, which shows that evaluating  $\langle \sigma, \text{foo} := 3 ; \text{foo} * \text{bar} \rangle$  using a store  $\sigma$  such that  $\sigma(\text{bar}) = 7$  yields  $\sigma' = \sigma[\text{foo} \mapsto 3]$  and 21 as a result:

$$\frac{\frac{\frac{}{\langle \sigma, 3 \rangle \Downarrow \langle \sigma, 3 \rangle} \text{INT} \quad \frac{\frac{}{\langle \sigma', \text{foo} \rangle \Downarrow \langle \sigma', 3 \rangle} \text{VAR} \quad \frac{}{\langle \sigma', \text{bar} \rangle \Downarrow \langle \sigma', 7 \rangle} \text{VAR}}{\langle \sigma', \text{foo} * \text{bar} \rangle \Downarrow \langle \sigma', 21 \rangle} \text{MUL}}{\langle \sigma, \text{foo} := 3 ; \text{foo} * \text{bar} \rangle \Downarrow \langle \sigma', 21 \rangle} \text{ASSGN}}$$

A closer look to this structure reveals the relation between small step and large-step evaluation: a depth-first traversal of the large-step proof tree yields the sequence of one-step transitions in small-step evaluation.

## 2 Equivalence of semantics

A natural question to ask is whether the small-step and large-step semantics are equivalent. The next theorem answers this question affirmatively.

**Theorem** (Equivalence of semantics). *For all expressions  $e$ , stores  $\sigma$  and  $\sigma'$ , and integers  $n$  we have:*

$$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle \text{ if and only if } \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$$

To streamline the proof, we will work with the following definition of the multi-step relation:

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle} \text{REFL}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \quad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle} \text{TRANS}$$

*Proof sketch.* We show each direction separately.

$\implies$ : We want to prove that the following property  $P$  holds for all expressions  $e \in \mathbf{Exp}$ :

$$P(e) \triangleq \forall \sigma, \sigma' \in \mathbf{Store}. \forall n \in \mathbf{Int}. \langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle \implies \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$$

We proceed by structural induction on  $e$ . We have to consider each of the possible axioms and inference rules for constructing an expression.

**Case  $e = x$ :** Assume that  $\langle \sigma, x \rangle \Downarrow \langle \sigma', n \rangle$ . That is, there is some derivation in the large-step operational semantics whose conclusion is  $\langle \sigma, x \rangle \Downarrow \langle \sigma', n \rangle$ . There is only one rule whose conclusion matches the configuration  $\langle \sigma, x \rangle$ : the large-step rule VAR. Thus, we have  $n = \sigma(x)$  and  $\sigma' = \sigma$ . By the small-step rule VAR, we also have  $\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle$ . By the REFL and TRANS rules, we conclude that  $\langle \sigma, x \rangle \rightarrow^* \langle \sigma, n \rangle$ , which finishes the case.

**Case  $e = n$ :** Assume that  $\langle \sigma, n \rangle \Downarrow \langle \sigma', n' \rangle$ . There is only one rule whose conclusion matches  $\langle \sigma, n \rangle$ : the large-step rule INT. Thus, we have  $n' = n$  and  $\sigma' = \sigma$  and so  $\langle \sigma, n \rangle \rightarrow^* \langle \sigma, n \rangle$  by the REFL rule.

**Case  $e = e_1 + e_2$ :** This is an inductive case. We want to prove that if  $P(e_1)$  and  $P(e_2)$  hold, then  $P(e)$  also holds. Let's write out  $P(e_1)$ ,  $P(e_2)$ , and  $P(e)$  explicitly.

$$\begin{aligned} P(e_1) &= \forall n, \sigma, \sigma'. \langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n \rangle \implies \langle \sigma, e_1 \rangle \rightarrow^* \langle \sigma', n \rangle \\ P(e_2) &= \forall n, \sigma, \sigma'. \langle \sigma, e_2 \rangle \Downarrow \langle \sigma', n \rangle \implies \langle \sigma, e_2 \rangle \rightarrow^* \langle \sigma', n \rangle \\ P(e) &= \forall n, \sigma, \sigma'. \langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma', n \rangle \implies \langle \sigma, e_1 + e_2 \rangle \rightarrow^* \langle \sigma', n \rangle \end{aligned}$$

Assume that  $P(e_1)$  and  $P(e_2)$  hold. Also assume that there exist  $\sigma, \sigma'$  and  $n$  such that  $\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma', n \rangle$ . We need to show that  $\langle \sigma, e_1 + e_2 \rangle \rightarrow^* \langle \sigma', n \rangle$ .

We assumed that  $\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma', n \rangle$ . This means that there is some derivation whose conclusion is  $\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma', n \rangle$ . By inspection, we see that only one rule has a conclusion of this form: the ADD rule. Thus, the last rule used in the derivation was ADD and it must be the case that  $\langle \sigma, e_1 \rangle \Downarrow \langle \sigma'', n_1 \rangle$  and  $\langle \sigma'', e_2 \rangle \Downarrow \langle \sigma', n_2 \rangle$  hold for some  $n_1$  and  $n_2$  with  $n = n_1 + n_2$ .

By the induction hypothesis  $P(e_1)$ , as  $\langle \sigma, e_1 \rangle \Downarrow \langle \sigma'', n_1 \rangle$ , we must have  $\langle \sigma, e_1 \rangle \rightarrow^* \langle \sigma'', n_1 \rangle$ . Likewise, by induction hypothesis  $P(e_2)$ , we have  $\langle \sigma'', e_2 \rangle \rightarrow^* \langle \sigma', n_2 \rangle$ . By Lemma 1 below, we have,

$$\langle \sigma, e_1 + e_2 \rangle \rightarrow^* \langle \sigma'', n_1 + e_2 \rangle,$$

and by another application of Lemma 1 we have:

$$\langle \sigma'', n_1 + e_2 \rangle \rightarrow^* \langle \sigma', n_1 + n_2 \rangle$$

Then, using the small-step ADD rule and the multi-step TRANS rule, we have:

$$\frac{\frac{n = n_1 + n_2}{\langle \sigma', n_1 + n_2 \rangle \rightarrow \langle \sigma', n \rangle} \text{ ADD} \quad \frac{}{\langle \sigma', n \rangle \rightarrow^* \langle \sigma', n \rangle} \text{ REFL}}{\langle \sigma', n_1 + n_2 \rangle \rightarrow^* \langle \sigma', n \rangle} \text{ TRANS}$$

Finally, by two applications of Lemma 2, we obtain  $\langle \sigma, e_1 + e_2 \rangle \rightarrow^* \langle \sigma', n \rangle$ , which finishes the case.

**Case  $e = e_1 * e_2$ .** Similar to case for  $e_1 + e_2$  above.

**Case  $e = x := e_1; e_2$ .** Omitted. Try it as an exercise.

$\Leftarrow$ : We proceed by induction on the derivation of  $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$  with a case analysis on the last rule used.

**Case REFL:** Then  $e = n$  and  $\sigma' = \sigma$ . We immediately have  $\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle$  by the large-step rule INT.

**Case TRANS:** Then  $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$  and  $\langle \sigma'', e'' \rangle \rightarrow^* \langle \sigma', n \rangle$ . In this case, the induction hypothesis gives  $\langle \sigma'', e'' \rangle \Downarrow \langle \sigma', n \rangle$ . The result follows from Lemma 3 below.

□

**Lemma 1.** If  $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$ , then the following hold:

- $\langle \sigma, e + e_2 \rangle \rightarrow^* \langle \sigma', n + e_2 \rangle$
- $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
- $\langle \sigma, n_1 + e \rangle \rightarrow^* \langle \sigma', n_1 + n \rangle$
- $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
- $\langle \sigma, x := e; e_2 \rangle \rightarrow^* \langle \sigma', x := n; e_2 \rangle$

*Proof.* Omitted; try it as an exercise.

□

**Lemma 2.** If  $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$  and  $\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle$ , then  $\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle$ .

*Proof.* Omitted; try it as an exercise.

□

**Lemma 3.** If  $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$  and  $\langle \sigma'', e'' \rangle \Downarrow \langle \sigma', n \rangle$ , then  $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$ .

*Proof.* Omitted; try it as an exercise.

□