CS 4110

Programming Languages & Logics

Lecture 27
Recursive Types

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Many languages support data types that refer to themselves:

Java

```
class Tree {
    Tree leftChild, rightChild;
    int data;
}
```

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class Tree {
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OCaml

```
type tree = Leaf | Node of tree * tree * int
```

$$tree \triangleq unit + int \times tree \times tree$$

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class Tree {
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OCaml

```
type tree = Leaf | Node of tree * tree * int
```

λ-calculus?

$$\mathit{tree} = \mathbf{unit} + \mathbf{int} \times \mathit{tree} \times \mathit{tree}$$

Recursive Type Equations

We would like **tree** to be a solution of the equation:

$$\alpha = \textbf{unit} + \textbf{int} \times \alpha \times \alpha$$

However, no such solution exists with the types we have so far...

Unwinding Equations

We could *unwind* the equation:

$$\alpha = \textbf{unit} + \textbf{int} \times \alpha \times \alpha$$

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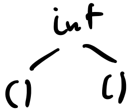
$$\begin{aligned}\alpha &= \textbf{unit} + \textbf{int} \times \alpha \times \alpha \\ &= \textbf{unit} + \textbf{int} \times \\ &\qquad (\textbf{unit} + \textbf{int} \times \alpha \times \alpha) \times \\ &\qquad (\textbf{unit} + \textbf{int} \times \alpha \times \alpha)\end{aligned}$$



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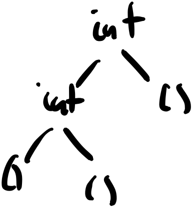

$$
\begin{aligned}
\alpha &= \textbf{unit} + \textbf{int} \times \alpha \times \alpha \\
&= \textbf{unit} + \textbf{int} \times \\
&\qquad (\textbf{unit} + \textbf{int} \times \alpha \times \alpha) \times \\
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&= \cdots
\end{aligned}
$$

If we take the limit of this process, we have an infinite tree.

Infinite Types

Think of this as an infinite labeled graph whose nodes are labeled with the type constructors $\times$, $+$, **int**, and **unit**.

This infinite tree is a solution of our equation, and this is what we take as the type **tree**.

μ Types

We'll specify potentially-infinite solutions to type equations using a finite syntax based on the *fixed-point type constructor* μ.

$$\mu\alpha.\,\tau$$

$$\alpha = \tau$$

$$\alpha = \text{unit} + \text{int} \times \alpha \times \alpha$$

μ Types

We'll specify potentially-infinite solutions to type equations using a finite syntax based on the *fixed-point type constructor* μ.

$$\mu\alpha.\,\tau$$

Here's a **tree** type satisfying our original equation:

$$\textbf{tree} \triangleq \mu\alpha.\,\textbf{unit} + \textbf{int} \times \alpha \times \alpha.$$

Static Semantics (Equirecursive)

We'll define two treatments of recursive types. With *equirecursive types*, a recursive type is equal to its unfolding:

$\mu\alpha.\,\tau$ is a solution to $\alpha = \tau$, so:

$$\mu\alpha.\,\tau = \tau\{\mu\alpha.\,\tau/\alpha\}$$

$tree \triangleq \mu\alpha.\,unit + int \times \alpha \times \alpha$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash inr_{\tau_1+\tau_2}\, e : \tau_1 + \tau_2}$$

$\vdash inr(1, inl\,(), inl\,()) : \mu\alpha.\,unit + int \times \alpha \times \alpha$

μ-elim

$\vdash inr(1, inl\,(), inl\,()) : unit + int \times \ldots \times \ldots$

μ-intro

$\vdash inr(1, inl\,(), inl\,()) : \mu\alpha.\,unit + int \times \alpha \times \alpha = tree$

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$$\mu\alpha.\,\tau = \tau\{\mu\alpha.\,\tau/\alpha\}$$

Two typing rules let us switch between folded and unfolded:

$$\frac{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}{\Gamma \vdash e : \mu\alpha.\,\tau}\ \mu\text{-INTRO}$$

$$\frac{\Gamma \vdash e : \mu\alpha.\,\tau}{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}\ \mu\text{-ELIM}$$

Isorecursive Types

Alternatively, *isorecursive types* avoid infinite type trees.

The type $\mu\alpha.\,\tau$ is distinct but transformable to and from $\tau\{\mu\alpha.\,\tau/\alpha\}$.

fold : $\tau\{\mu\alpha.\,\tau/\alpha\} \to \mu\alpha.\,\tau$

unfold : $\mu\alpha.\,\tau \to \tau\{\mu\alpha.\,\tau/\alpha\}$

$\vdash$ fold(inl ()) : tree

unit + ...

Isorecursive Types

Alternatively, *isorecursive types* avoid infinite type trees.

The type $\mu\alpha.\,\tau$ is distinct but transformable to and from $\tau\{\mu\alpha.\,\tau/\alpha\}$.

Converting between the two uses explicit **fold** and **unfold** operations:

$$
\begin{array}{rcl}
\textbf{unfold}_{\mu\alpha.\,\tau} & : & \mu\alpha.\,\tau \;\rightarrow\; \tau\{\mu\alpha.\,\tau/\alpha\} \\
\textbf{fold}_{\mu\alpha.\,\tau} & : & \tau\{\mu\alpha.\,\tau/\alpha\} \;\rightarrow\; \mu\alpha.\,\tau
\end{array}
$$

$$
\frac{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}{\Gamma \vdash \text{fold } e : \mu\alpha.\,\tau}\;\mu\text{-intro}
$$

Static Semantics (Isorecursive)

The typing rules introduce and eliminate μ-types:

$$\frac{\Gamma \vdash e : \tau\{\mu\alpha.\,\tau/\alpha\}}{\Gamma \vdash \textbf{fold}\; e : \mu\alpha.\,\tau}\ \mu\text{-INTRO}$$

$$\frac{\Gamma \vdash e : \mu\alpha.\,\tau}{\Gamma \vdash \textbf{unfold}\; e : \tau\{\mu\alpha.\,\tau/\alpha\}}\ \mu\text{-ELIM}$$

$e' \xrightarrow{*}$ unfold (fold e) $\longrightarrow$

Dynamic Semantics

We also need to augment the operational semantics:

~~fold (unfold e) ⤳ e~~

$$\frac{}{\textbf{unfold}\ (\textbf{fold}\ e) \rightarrow e}$$

Intuitively, to access data in a recursive type $\mu\alpha.\,\tau$, we need to **unfold** it first. And the only way that values of type $\mu\alpha.\,\tau$ could have been created is via **fold**.

Example

Here's a recursive type for lists of numbers:

$$\textbf{intlist} \triangleq \mu\alpha.\, \textbf{unit} + \textbf{int} \times \alpha.$$

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$$\textbf{intlist} \triangleq \mu\alpha.\, \textbf{unit} + \textbf{int} \times \alpha.$$

Here's how to add up the elements of an **intlist**:

$$
\begin{array}{l}
\text{let sum} = \\
\quad \text{fix}\,(\lambda f : \textbf{intlist} \to \textbf{intlist} \\
\qquad\quad \lambda l : \textbf{intlist}.\ \text{case}\ \textbf{unfold}\ \ell\ \text{of} \\
\qquad\qquad\qquad (\lambda u : \textbf{unit}.\ 0) \\
\qquad\qquad\quad\ \mid (\lambda p : \textbf{int} \times \textbf{intlist}.\ (\#1\,p) + f\,(\#2\,p)))
\end{array}
$$

Encoding Numbers

Recursive types let us encode the natural numbers!

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A natural number is either 0 or the successor of a natural number:

$$\textbf{nat} \triangleq \mu\alpha.\, \textbf{unit} + \alpha$$

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$$\begin{aligned} \textbf{nat} &\triangleq \mu\alpha.\, \textbf{unit} + \alpha \\ 0 &\triangleq \textbf{fold}\,(\mathsf{inl}_{\textbf{unit}+\textbf{nat}}\,())\\ 1 &\triangleq \textbf{fold}\,(\mathsf{inr}_{\textbf{unit}+\textbf{nat}}\,0)\\ 2 &\triangleq \textbf{fold}\,(\mathsf{inr}_{\textbf{unit}+\textbf{nat}}\,1),\\ &\vdots \end{aligned}$$

Encoding Numbers

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A natural number is either 0 or the successor of a natural number:

$$\begin{aligned} \textbf{nat} &\triangleq \mu\alpha.\, \textbf{unit} + \alpha \\ 0 &\triangleq \textbf{fold}\,(\text{inl}_{\textbf{unit}+\textbf{nat}}\,()) \\ 1 &\triangleq \textbf{fold}\,(\text{inr}_{\textbf{unit}+\textbf{nat}}\,0) \\ 2 &\triangleq \textbf{fold}\,(\text{inr}_{\textbf{unit}+\textbf{nat}}\,1), \\ &\vdots \end{aligned}$$

The successor function has type $\textbf{nat} \to \textbf{nat}$:

$$(\lambda x : \textbf{nat}.\, \textbf{fold}\,(\text{inr}_{\textbf{unit}+\textbf{nat}}\,x))$$

Self-Application and Ω

Recall Ω defined as:

$$\omega \triangleq \lambda x.\, x\, x \qquad\qquad \Omega \triangleq \omega\, \omega.$$

Ω was impossible to type... until now!

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x is a function. Let's say it has the type $\sigma \to \tau$. $= \alpha$

$\lambda x : \mu\alpha.\, \alpha \to \tau$

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x is used as the argument to this function, so it must have type σ.

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x is a function. Let's say it has the type $\sigma \to \tau$.

x is used as the argument to this function, so it must have type σ.

So let's write a type equation:

$$\sigma = \sigma \to \tau$$

$\mu\alpha.\, \alpha \to \tau$

$(\lambda x : \mu\alpha.\alpha \to \tau.\ (\text{unfold } x)\ x) : (\mu\alpha.\, \alpha \to \tau) \to \tau$

Self-Application and Ω

Putting these pieces together, the fully typed ω term is:

$$\omega \triangleq \lambda x : \mu\alpha.\, (\alpha \to \tau).\, (\textbf{unfold}\, x)\, x$$

$\mu\alpha.\,(\alpha \to \tau) \to \tau$

$\text{fold}\ \omega : \mu\alpha.\,\alpha \to \tau$

Self-Application and Ω

Putting these pieces together, the fully typed ω term is:

$$\omega \triangleq \lambda x : \mu\alpha.\,(\alpha \rightarrow \tau).\,(\textbf{unfold}\; x)\; x$$

The type of ω is $(\mu\alpha.\,(\alpha \rightarrow \tau)) \rightarrow \tau$.

So the type of **fold** ω is $\mu\alpha.\,(\alpha \rightarrow \tau)$.

Self-Application and Ω

Putting these pieces together, the fully typed ω term is:

$$\omega \triangleq \lambda x : \mu\alpha.\,(\alpha \to \tau).\,(\textbf{unfold}\ x)\ x$$

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So the type of **fold** ω is $\mu\alpha.\,(\alpha \to \tau)$.

Now we can define $\Omega = \omega\ (\textbf{fold}\ \omega)$. It has type τ.

Self-Application and Ω

We can even write ω in OCaml:

```
# type u = Fold of (u -> u);;
type u = Fold of (u -> u)
# let omega = fun x -> match x with Fold f -> f x;;
val omega : u -> u = <fun>
# omega (Fold omega);;
...runs forever until you hit control-c
```

Encoding λ-Calculus

With recursive types, we can type everything in the untyped lambda calculus!

$$U \to U = U$$

$$\mu\alpha.\, \alpha \to \alpha$$

Encoding λ-Calculus

With recursive types, we can type everything in the untyped lambda calculus!

Every λ-term can be applied as a function to any other λ-term. So let's define an "untyped" type:

$$U \triangleq \mu\alpha.\, \alpha \to \alpha$$

$$[\![e_1 \ e_2]\!] \triangleq (\mathrm{unfold}\ e_1)\ e_2 \qquad (\mathrm{unfold}\ e_1 : U \to U)$$

$$[\![\lambda x.\, e]\!] \triangleq \mathrm{fold}\,(\lambda x : U.\, e)$$

$$[\![x]\!] \triangleq x$$

Encoding λ-Calculus

With recursive types, we can type everything in the untyped lambda calculus!

tree α = Leaf | α × tree × tree

Every λ-term can be applied as a function to any other λ-term. So let's define an "untyped" type:

$$U \triangleq \mu\alpha.\, \alpha \to \alpha$$

λx: tree. e

The full translation is:

⊢ e : int

$$\begin{aligned} [\![x]\!] &\triangleq x \\ [\![e_0\, e_1]\!] &\triangleq (\textbf{unfold}\ [\![e_0]\!])\ [\![e_1]\!] \\ [\![\lambda x.\, e]\!] &\triangleq \textbf{fold}\ \lambda x : U.\, [\![e]\!] \end{aligned}$$

λα. μα. ...

Every untyped term maps to a term of type U.