NP and Nondeterminism

More traditional way of viewing NP:

- Imagine a *nondeterministic* algorithm, where next step is not determined.
  - E.g. choose a random number n and set $x = n$
- L is in NP if there is a nondeterministic algorithm A that runs in polynomial time such that
  - if $x \in L$, some computation accepts (returns 1)
  - if $x \notin L$, no computation accepts
- "runs in polynomial time" means exists c such that all computations on input x run in time $O(|x|^c)$.
  - Because of the nondeterminism, different computations on input x may have different running times.

Connection to previous definition:

- if there's a verification algorithm, can convert it to a nondeterministic polynomial algorithm:
  - nondeterministically try all possible verification strings y such that $|y| = O(|x|^c)$
  - Can do this in PTIME with branching
- Conversely, if there's a nondeterministic algorithm, can convert it to a verification algorithm:
  - y describes the choices made along a given branch

NP, Co-NP, and PTIME

L is in co-NP if $\overline{L}$ is in NP:

Examples:

- L is the set of encodings of graphs that do *not* have Hamiltonian paths.

Major questions of complexity theory:

- Does P = NP?
  - Probably not, but no proof yet
- If P = NP, then there are PTIME algorithms for lots of problems that we don't know how to do efficiently yet
  - E.g., factoring, scheduling, bin-packing, ...
- Does P = co-NP?
  - Since P is closed under complementation, this is true iff P = NP (see homework)
- Does NP = co-NP?
- Does P = NP $\cap$ co-NP?
  - We can't answer any of these questions (yet)
  - Solving them gets you a Turing award ...

The little we know:

- P $\subseteq$ NP/co-NP $\subseteq$ PSPACE $\subseteq$ EXPTIME
- P $\neq$ PSPACE

Reducibility

Key idea in complexity theory: *reducibility*

- Making precise the well-known mathematical idea of reducing one problem to another
- Idea: If you can reduce L_1 to L_2, then if you have an efficient algorithm to decide L_2, then you get an efficient algorithm to decide L_1

Formal definition:

$L_1 \subseteq \Sigma^*$ is *polynomial-time reducible to* $L_2 \subseteq (\Sigma')^*$ if there is a polynomial time computable function $f : \Sigma^* \to (\Sigma')^*$ such that $x \in L_1$ iff $f(x) \in L_2$.

Lemma 1: If $L_2 \in$ P and $L_1 \leq_P L_2$, then $L_1 \in P$.

Proof: Suppose A_2 is a PTIME algorithm that decides L_2, and f reduces L_1 to L_2

- $x \in L_1$ iff $f(x) \in L_2$

Let $A_1(x) = A_2(f(x))$.

- A_1 is PTIME, since A_2 and f are.
- $x \in L_1$ iff $f(x) \in L_2$ iff $A_1(x) = A_2(f(x)) = 1$.

Lemma 2: Reduction is transitive: If $L_1 \leq_P L_2$ and $L_2 \leq_P L_3$, then $L_1 \leq_P L_3$.

Proof: Suppose f reduces L_1 to L_2, g reduces L_2 to L_3:

- $x \in L_1$ iff $f(x) \in L_2$
- $x \in L_2$ iff $g(x) \in L_3$.

Then $x \in L_1$ iff $g(f(x)) \in L_3$.

$g \circ f$ is PTIME computable.

Therefore $L_1 \leq_P L_3$ (using $g \circ f$)

NP-Completeness

A language L is *NP-complete* if

1. L is in NP and
2. L is *NP hard* – i.e., L is the "hardest" NP problem:
   - every language L' in NP can be reduced to L
   - If $L' \in NP$, then $L' \leq_P L$

Theorem: If any NP-complete language is in P, then every language in NP is in P.

Proof: Suppose that L is NP-complete, and L is in P. If $L' \in$ NP, then $L' \leq_P L$. Therefore L' is in P.

There are *thousands* of known NP-complete languages.

- See Garey and Johnson (1979) for the classic compendium

We haven't found any PTIME algorithm for any of them yet.

Proving a Language is NP-complete

General strategy for proving language L is NP-complete:

- Show L is in NP (usually easy)
- Reduce a known NP-complete problem L' to L.
  - That is, show that $L' \leq_P L$
  - This means L is *NP-hard*
    - This is because $\leq_P$ is transitive
    - If L'' is in NP, $L'' \leq_P L'$
    - Since $L' \leq_P L$, it follows that $L'' \leq_P L$.

Thus, it helps to have a core set of NP-complete problems to start with.

Getting off the ground is hard:

- How do you prove that *every* language in NP can be reduced to a particular language L?

For this we need a model of computation.

Turing Machines

A Turing machine (TM) can be thought of as an infinite tape, where a head can write 0s and 1s, together with some instructions for what to write.

- initially the tape has the input written on it.

Key question:

- How are instructions described?
  - i.e., what is the programming language?
- Idea: there is a finite set of *states*
- In a given state, the head can
  - read the symbol on the tape cell under it,
  - write a symbol (0/1) on the tape cell under it,
  - move one step left or one step right,
- Then the TM can change to a new state.
  - The new state depends on the old state and the symbol read.
  - There may be more than one possible next state (nondeterminism).

This may not like a very powerful model of computation, but ...

- Every program in a standard programming language (Java, C) corresponds to some TM

To show that a language L is NP-hard, we have to show that for every language L' in NP, there is a function $f_{L'}$ such that $x \in L'$ iff $f_{L'}(x) \in L$.

- Idea: since $L' \in$ NP, there is a TM $M_{L'}$ that outputs 1 on input x iff $x \in L$
- $f_{L'}(x)$ simulates the computation of $M_{L'}$ on x

Satisfiability: the canonical NP-complete problem

Propositional logic:

- Start with a set of primitive propositions $\{p_1, \ldots, p_n\}$.
- Form more complicated formulas by closing off under conjunction ($\wedge$) and negation ($\neg$)

Typical formula: $\neg(p_1 \wedge \neg p_2) \wedge (p_2 \wedge \neg p_1)$.
Standard abbreviation: $p \vee q$ is an abbreviation for $\neg(\neg p \wedge \neg q)$.

Given a formula, we want to decide if it is true or false.

- The truth or falsity of a formula depends on the truth or falsity of the primitive propositions that appear in it. We use *truth tables* to describe how the basic connectives ($\neg$, $\wedge$) work.

Truth Tables

For $\neg$:

p	$\neg p$
T	F
F	T

For $\wedge$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

For $\vee$:

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q) = p \vee q$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

This means that $\vee$ is *inclusive* or, not *exclusive* or.

Equivalence

Two formulas are *equivalent* if the same truth assignments make them true.

Examples:

- Distribution Laws:
  - $p \wedge (q_1 \vee q_2)$ is equivalent to $(p \wedge q_1) \vee (p \wedge q_2)$
  - $p \vee (q_1 \wedge q_2)$ is equivalent to $(p \vee q_1) \wedge (p \vee q_2)$
- DeMorgan's Laws
  - $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$
  - $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$

How do you check if two formulas are equivalent?

- Fill in the truth tables for both.

Satisfiability

Is $(p_1 \vee p_2) \wedge (\neg p_2 \vee p_3) \wedge (\neg p_3 \vee p_1)$ satisfiable?

- Is there a truth assignment to the primitive propositions that makes this formula true?
  - Yes: $p_1 \leftarrow T$, $p_2 \leftarrow T$, $p_3 \leftarrow T$
- How about $(p_1 \vee p_2) \wedge (\neg p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_1)$?
  - $p_1 \leftarrow T$, $p_2 \leftarrow T$, $p_3 \leftarrow T$ doesn't work.
  - $p_1 \leftarrow T$, $p_2 \leftarrow F$, $p_3 \leftarrow F$ does.
- How about $(p_1 \vee p_2) \wedge (\neg p_2 \vee p_3) \wedge (\neg p_3 \vee \neg p_1) \wedge \neg p_1$?
  - Nothing works ...

In general, you can tell if a formula is satisfiable by guess a truth assignment, and verifying that it works.

- The truth assignment is a certificate ...

Satisfiability is also NP-hard ....

Idea of proof:

- Start with a language L' in NP and input x
- Since L' is in NP, there exists c, k, and a (nondeterministic) TM $M_{L'}$ such that $M_{L'}$ accepts L' using at most $c|x|^k$ steps on input x
- Construct formula $\varphi_{x,L'}$ that is satisfiable iff $x \in L'$
- Want $|\varphi_{x,L'}|$ to be $O(|x|^{2k})$
- Then $f_{L'}(x) = \varphi_{x,L'}$

Main ideas of construction

- $M_{L'}$ uses at most $c|x|^k$ cells on the tape
- Have propositions $p_{0,i,t}$, $p_{1,i,t}$, $p_{b,i,t}$, $i, t = 1, \ldots c|x|^k$
  - cell i has a $0/1/b$ (b for blank) at step t
- Part of $\varphi_{x,L'}$ says that exactly one of $p_{0,i,t}$, $p_{1,i,t}$, $p_{b,i,t}$ holds at each time t

$$(p_{0,i,t} \vee p_{1,i,t} \vee p_{b,i,t}) \wedge$$
$$\neg(p_{0,i,t} \wedge p_{1,i,t}) \wedge \neg(p_{0,i,t} \wedge p_{b,i,t}) \wedge \neg(p_{1,i,t} \wedge p_{b,i,t})$$

- Have propositions $p_{h,i,t}$, $i, t = 1, \ldots, c|x|^k$
  - The head is in position i at time t

- Exactly one of $p_{h,1,t}, \ldots, p_{h,c|x|^k,t}$ holds (for all t)
- $p_{h,1,1}$ holds
  - The tape is initially at the far left
- If $x = x_1 \ldots x_k$, then $p_{x_1,1,1} \wedge p_{x_2,2,1} \wedge \ldots \wedge p_{x_k,k,1} \wedge p_{b,k+1,1} \wedge p_{b,c|x|^k,1}$ holds
  - x is written out initially at the far left of the tape, followed by blanks.
- Similarly, can say that at time $c|x|^k$, there is a 1 at the far left, followed by blanks
  - $M_{L'}$ accepts x
- The hard part is to write the part of the formula that captures the step-by-step operation of $M_{L'}$.
  - Need proposition that talk about the current state of $M_{L'}$ and how it changes

Bottom line: We can simulate TMs that run in nondeterministic polynomial time using propositional logic.

- Satisfiability is NP complete!
- This was the first problem proved NP complete (by Steve Cook)
- Validity is co-NP complete

3-CNF Satisfiability

A *literal* is a primitive proposition or its negation:

- p or $\neg p$

A clause is a disjunction of distinct literals:

- $p_1 \vee p_3 \vee \neg p_7 \vee p_2 \vee \neg p_5$

A formula is in CNF (*conjunctive normal form*) if it is a conjunction of clauses

$$(p_1 \vee \neg p_3) \wedge (p_1 \vee p_5 \vee \neg p_2 \vee p_7) \wedge (p_3 \vee \neg p_5)$$

A formula is in k-CNF if each clause has exactly k literals.

Theorem: The satisfiability problem for 2-CNF formulas is in P.

Theorem: The satisfiability problem for 3-CNF formulas in NP-complete.

Proof: It's clearly in NP. To show that it's NP-hard, it suffices to show that the satisfiability of an arbitrary formula φ can be reduced in polynomial to the satisfiability of a 3-CNF formula φ'.

Three steps:

Step 1:

- Write a binary parse tree for φ,
  - internal nodes are labeled with $\neg$, $\wedge$, and $\vee$
  - leaves are labeled with literals
  - An internal node represents a subformula of φ
  - Introduce a new primitive proposition q for each internal node
  - Write formula that says that q characterizes the formula at that node.
    - If internal node is $\neg$ and successor is labeled by q',
      $$(q \wedge \neg q') \vee (\neg q \wedge q')$$
    - If internal node is $\wedge$ and successors are q_1 and q_2:
      $$(q \wedge q_1 \wedge q_2) \vee (\neg q \wedge \neg(q_1 \wedge q_2))$$
- Let φ' be the conjunction of these formulas.
  - Not hard to show that φ' is satisfiable iff φ is satisfiable

Step 2: Convert φ' to an equivalent CNF formula, using various equivalences, where each clause has at most 3 literals:

- Using Distribution Laws, $(q \wedge \neg q') \vee (\neg q \wedge q')$ is equivalent to
  $$(q \vee \neg q) \wedge (q \vee q') \wedge (\neg q' \vee \neg q) \wedge (\neg q' \vee q')$$
- Using Distribution Laws and DeMorgan's Laws, can do the same for other clauses.
- (Actually, *every* formxla is equivalent to a CNF formula)

Step 3: Get an equi-satisfiable 3-CNF formula

- Replace a disjunct $p_1 \vee p_2$ by
  $$(p_1 \vee p_2 \vee q) \wedge (p_1 \vee p_2 \vee \neg q)$$
- The new formula is satisfiable iff the original was.