Matrix Chain Multiplication

The input to the following algorithm is $p = (p_0, \ldots, p_n)$, where $p_{i-1} \times p_i$ is the dimension of A_i.

- $s[i, j]$ is the best place to split the computation of $A_{i..j}$ to $A_{i..k}A_{k+1..j}$.

```
MATRIX-CHAIN-ORDER(p)
n ← length[p] − 1
for i ← 1 to n do
    m[i, j] ← 0
for l ← 2 to n do
    for i ← 1 to n − l + 1 do
        j ← i + l − 1
        m[i, j] ← ∞
        for k ← i to j − 1 do
            q ← m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
               then m[i, j] ← q
                    s[i, j] ← k
return m and s
```

Running time: $O(n^3)$

- Key point: the same information ($m[i, j]$) gets reused over and over.

Computing an optimal solution

MATRIX-CHAIN-ORDER computes the best place to split and the optimal number of scalar multiplications.

- From $s[i, j]$, it's easy to compute how to multiply

```
M-CHAIN-MULTIPLY(A, s, i, j)
if j > i
   then X ← M-CHAIN-MULTIPLY(A, s, i, s[i, j])
        Y ← M-CHAIN-MULTIPLY(A, s, s[i, j] + 1, j)
        return MATRIX-MULTIPLY(X, Y)
   else return A_i
```

Get the right answer by calling M-CHAIN-MULTIPLY$(A, s, 1, n)$

Longest Common Subsequence

Given two sequences, we want to find there longest common subsequence. This is a problem that comes up, for example, in gene sequencing (if we want to compare to genomes).

Formally, if $Z = (z_1, \ldots, z_k)$ is a subsequence of $X = (x_1, \ldots, x_m)$ if there exist $i_1, \ldots, i_k$ such that $i_1 < \ldots < i_k$ and $z_j = x_{i_j}$.

Example: The longest common subsequence of (A, A, B, C, A, A, D, A) and $(A, C, B, C, A, B, D, C, A)$ is (A, B, C, A, D, A).

- There can be at most 3 A's in the lcs, so this is the best we can do.

The brute force approach to finding LCS of X and Y is to consider all subsequences of X and see which ones are subsequences of Y.

- The number of subsequences of X is exponential in length(X).

We can do better using dynamic programming.

Characterizing an LCS

Given a sequence $X = (x_1, \ldots, x_m)$, if $i \leq m$, let $X_i = (x_1, \ldots, x_i)$.

Theorem: Suppose that $Z = (z_1, \ldots, z_k)$ is an lcs for $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_n)$.

1. If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an lcs for X_{m-1} and Y_{n-1}.
2. If $x_m \neq y_n$ and $z_k \neq x_m$, then Z is an lcs for X_{m-1} and Y_n.
3. If $x_m \neq y_n$ and $z_k \neq y_n$, then Z is an lcs for X and Y_{n-1}.

Therefore, an lcs for X and Y contains within it an lcs for two smaller sequences.

- We can find LCS(X, Y) by first finding LCS(X_i, Y_j) for all the prefixes of X and Y.

Solving LCS Recursively

Let $c[i, j]$ the length of an lcs of X_i and Y_j.

$$c[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } i, j > 0,\ x_i = y_j \\ \max(c[i-1, j], c[i, j-1]) & \text{if } i, j > 0,\ x \neq y_j \end{cases}$$

LCS-Length(X, Y)

```
n ← length[X]
m ← length[Y]
for i ← 1 to m do
      c[i, 0] ← 0
for j ← 0 to n do
      c[0, j] ← 0
for i ← 1 to m do
      for j ← 1 to n do
            if x_i = y_j
               then c[i, j] ← c[i − 1, j − 1] + 1
               else c[i, j] ← max(c[i − 1, j], c[i, j − 1])
return c
```

Running time (and space): $O(nm)$

Printing out an LCS

Print-LCS(c, X, i, j)

```
if i = 0 or j = 0
   then return
if c[i − 1, j] = c[i, j]
   then Print-LCS(c, X, i − 1, j)
   else if c[i, j − 1] = c[i, j]
             then Print-LCS(c, X, i, j − 1)
             else Print-LCS(c, X, i − 1, j − 1)
                  print x_i
```


Greedy Algorithms

One approach to an optimization problem: make the choice that currently looks best.

- Sometimes this greedy approach is a bad idea
  - you can get caught in a trap
- Other times it works remarkably well.

Kruskal's algorithm for MST can be viewed as a greedy algorithm:

- Choose the edge of least weight that buys you something

So can Prim's algorithm:

- Choose the edge of least weight that extends the current tree and buys you something.

And so can Dijkstra's algorithm:

- Choose the node not yet chosen which is closest to the source.

Activity Selection

Suppose that we have a set $S = \{1, \dots, n\}$ of proposed *activities* that need to use the same resource

- only one can be active at a time
  - example: scheduling classes in a lecture hall
- Activity i has a *start time* s_i and a *finish time* f_i.

Problem: choose the maximum set of mutually compatible activities

- Don't want activities whose start-finish times overlap

Basic idea: keep choosing an activity as long as it's compatible with the ones you've already chosen.

- The actual algorithm suggests a particular way to choose.

Order the activities by increasing finish time:

$$f_1 \leq f_2 \leq \ldots \leq f_n$$

- This pre-processing step takes time $O(n \log n)$

Assume the algorithm gets as input the sequence s of start times and the sequence f of finish times (in sorted order):

GREEDY-ACTIVITY-SELECTOR(s, f)

```
n ← length[s]
A ← {1}                [A consists of selected activities]
j ← 1                  [j is the last activity put into A]
for j ← 2 to n do
    if s_i ≥ f_j       [if it's safe to add i to A ...]
      then A ← A ∪ i
           j ← i
return A
```

Clearly this gives a set of compatible activities.

It's also efficient:

- After preprocesesing, run in time $\Theta(n)$.

But why is it correct?

Theorem: GREEDY-ACTIVITY-SELECTOR chooses a maximum set of mutually compatible activities.

Proof: By strong induction on n, the number of activies in S.

Base case: clearly OK if $S = 1$.

Inductive step: First show that there is a maximum set that includes activity 1 (the one with earliest finish time).

Let A be a maximum set and let k be the activity in A with earliest finish time.

- If $k = 1$, we're done.
- If not, let $B = A - \{k\} \cup \{1\}$. The activities in B must be mutually compatible
  - activity 1 can't overlap with anything, since its finish time is earlier than k's
- Thus, B is a maximum set that includes 1.

If A is a maximum set of mutually compatible activities in $S = \{1, \ldots, n\}$ and $1 \in A$, then $A - \{1\}$ is a maximum set of mutually compatible activities in $S' = \{i \in S : s_i \geq f_1\}$.

- S' consists of activities that start after 1 ends.

Now by induction, the algorithm produces a maximum set on S'.

- But the action of algorithm on S' is exactly the same as the action of the algorithm on S after choosing 1.

Greedy vs. Dynamic Programming

A greedy algorithm works only if making the greedy choice gives an optimal solution:

- That works in some cases, but not always.
- The hard part is often showing that it works

Example:

- The *0-1 knapsack problem*: there are n items
  - Item i has value v_i and weight w_i.

  You can put at most W pounds into a knapsack. Which items do you take?
  - For each item, you either take it or leave it (0-1)
- The *fractional knapsack problem*: same setup, but now you can take part of an item.
  - This means you have more flexibility

Key point:

- There's a greedy algorithm for the fractional knapsack problem, but not for the 0-1 knapsack problem

For the fractional knapsack problem:

- First sort the items by value/pound (v_i/w_i)
- Pick the most valuable items that you can fit in, then the next one, etc., until there's no more room.
- Then put in as much of the last item as you can to get to weight W.
  - This is OK since you can take fractions of an item.

This approach doesn't work for the 0-1 knapsack problem:

- Suppose there are three items and the knapsack can hold 50 pounds:
  - Item 1 weighs 10 lb. and is worth \$60
  - Item 2 weighs 20 lb. and is worth \$100
  - Item 3 weighs 30 lb. and is worth \$120
- Item 1 is the most valuable, but the optimal solution is $\{2, 3\}$.

You can use dynamic programming to solve the 0-1 knapsack problem.

Huffman Codes

Suppose you have a large file, where only 6 different characters appear

- Not all characters appear equally often
- How do we represent the characters so as to get greatest compression?
  - Compression is critical in transmitting data over a modem
  - There are *lots* of coding algorithms

Assume each character is represented as a binary string. Example:

$a = 000000$ $b = 000001$ ...
$z = 011010$ $, = 011011$ ...

Is this a good encoding?

- This is a *fixed-length* code: all characters encoded by a 6-bit code word
- It's a better idea to use a *variable-length* code
- Greater frequency $\Rightarrow$ shorter code word
  - Modern coding algorithms (based on Ziv-Lempel) adaptively choose length of code word

Prefix Code

If one code is a prefix of another, then decoding is harder

- if e is 0 and a is 01, when you see 0, is it an e or the beginning of an a.

It is best to assume a *prefix code*

- no codeword is the prefix of another codeword.

Decoding is simple with a prefix code:

- Keep running along string until you have a complete codeword, and continue
  - Note: this is a greedy decoding algorithm
- E.g., suppose $e = 0$, $a = 10$, $b = 110$
  - then $00110100 = eebae$