CS280, Spring 2000: Final Solutions

1. To show that $\sim$ is an equivalence relation, you have to show that it is reflexive, symmetric, and transitive. It is reflexive since $(m,n) \sim (m,n)$ (since $m+n = m+n$). To see that it is symmetric, suppose that $(m,n) \sim (k,l)$. Then $m+n = k+l$ so $(k,l) \sim (m,n)$. Finally, suppose that $(m,n) \sim (k,l)$ and $(k,l) \sim (a,b)$. Then $m+n = k+l$ and $k+l = a+b$. Since equality is transitive, it follows that $m+n = a+b$, so $(m,n) \sim (a,b)$.

2. It is easy to see that $s_1 = 2/1 = 2$, $s_2 = 2/s_1 = 1$, $s_3 = 2/s_2 = 2$. This should lead you to guess that the sequence is 1, 2, 1, 2, ...; i.e., $s_n = 1$ if n is odd and $s_n = 2$ if n is even. Let $P(n)$ be the statement that $s_n = 1$ if n is odd and 2 if n is even. We show $P(n)$ for all $n \geq 1$.

   Base case: $s_1 = 1$ by definition.

   Inductive step: Suppose $P(n)$ holds. We prove $P(n+1)$. There are two cases. If $n+1$ is odd, then n is even and, by the inductive hypothesis, $s_n = 2$. Then $s_{n+1} = 2/s_n = 1$. If $n+1$ is even, then n is odd, so $s_n = 1$. Then $s_{n_1} = 2/s_n = 2$. This proves $P(n+1)$.

3. (a) I prove that $n = m2^k$ is a loop invariant by induction on the number of iterations through the loop. Let $P(N)$ be the statement "if the program goes through the loop N times, then after then Nth time through, $n = m2^k$, $k = N$, and m is an positive integer." (I don't need the fact that $k = N$ or that m is a positive integer for part (a), but they're useful for part (b).) We prove $P(N)$ for all $N \geq 1$.

   Base case: Initially (when $N = 0$), $k = 0$ and $n = m$, so $P(0)$ holds.

   Inductive step: Suppose that $P(N)$ holds. We show that $P(N+1)$ holds. If the program does not go through the loop $N+1$ times, then there is nothing to prove. If it does, let k' and n' be the values of k and n after the Nth iteration. By the induction hypothesis, $n = m'2^{k'}$, $k' = N$, and m' is a positive integer. Since the program goes through the loop $N+1$ times, it must be the case that m' is even, $m = m'/2$, and $k = k'+1$. Thus, m is a positive integer, $k = N$, and $m2^k = (m'/2)2^{k'+1} = m'2^{k'} = n$. This proves $P(N+1)$.] [Comments: very few people did this right. A lot of people took the induction hypothesis $P(n)$ to be $n = 2^m k$. That doesn't make sense; n is constant. You definitely don't want to be prove $P(n)$ for all n. Rather you want to show that $n = 2^m k$ each time you go trhough the loop. So you have to make the N in the induction statement be the number of times through the loop.]

   (b) The best way to prove this formally is to take the induction statement $Q(n)$ to be that, when started with input n, the program terminates with m being odd. I prove $Q(n)$ for all n by strong induction. If $n = 1$, the program terminates without entering the loop, with $m = 1$. If $n > 1$ and odd, then the program terminates immediately with $n = m$. If n is even, then initially $m = n$. After going through the loop once, $m = n/2$. The program continues from this point just as it did if it starts with input $n/2$. By the induction hypothesis $P(n/2)$, when started with input $n/2$, the program terminates with m being odd, so we are done.

Here's an alternative proof, using the fact that part (a) showed that if the program goes through the loop N times, then after then Nth iteration, $n = m2^N$; i.e., $m = n/2^N$, and m is a positive integer. Thus, the program can go through the loop at most $\lfloor \log_2(n) \rfloor$ times. If m cannot be even when the program terminates, for then the program would have gone through the loop again.

[Comments: it's not enough to say that "eventually" m is odd if you keep dividing by 2. That's exactly what you have to prove. (Although partial credit was given if you said this. How much depended on what else you said.)

4. (a) The adjacency matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(b) Here is the table for Dijkstra's algorithm:

a	b	c	d	z	Vertex added to U
0	∞	∞	∞	∞	a
0	6	2	∞	∞	c
0	4	2	5	7	b
0	4	2	5	7	d
0	4	2	5	6	z

Thus, the shortest distance from a to z is 6.

5. It is *not* possible for an insect to crawl along the edges of a cube so as to travel along each edge exactly once. If it were possible, then there would be a Euclidean path on the cube. But each vertex of a cube has odd outdegree (in fact, outdegree 3), and we proved in class that a graph has a Euclidean path iff either 0 or two vertices on the graph have odd degree.

6. Let $G = (V, E)$. Here are two formal proofs:

(1) I prove by induction on k that $P(k)$ holds for $k \geq 0$, where $P(k)$ is the statement that either there is an elementary cycle of length $\leq k$ in G or there is a elementary path $(v_0, \ldots, v_k)$; i.e., one with repeated vertices. The base case is easy: just choose an arbitrary vertex v_0 and consider the path (v_0). Suppose that $P(k)$ holds. Then either there is an elementary cycle of length $\leq k$ in G, in which case $P(k+1)$ holds,

or there is an elementary path $(v_0, v_1, \ldots, v_k)$ with no repeated edges. Since v_k has degree 2, there must be a vertex v' other than v_{k-1} such that $(v, v') \in E$. If v' is one of $v_0, \ldots, v_{k-2}$, say v_j, then $(v_j, \ldots, v_k, v')$ is an elementary cycle in G of length $\leq k+1$. If v' is not in $(v_0, \ldots, v_k)$, then $(v_0, \ldots, v_k, v')$ is an elementary path of length $k+1$. This proves the inductive statement.

Suppose $|V| = n$. Since $P(n)$ holds, there must either be an elementary path in G of length n or a cycle of length $\leq n$. Since an elementary path of length n that is not a cycle has $n+1$ different vertices, there can't be an elementary path in G of length n that is not a cycle. Thus, G must have a cycle (of length $\leq n$).

(2) This proof is simpler to do, but perhaps harder to find. Let $P(n)$ be the statement that if G has n vertices each of which has degree 2, then G has an elementary cycle. I prove $P(n)$ by induction on n. If $n = 1$, then the only way for the one vertex to have degree 2 is if it has a self-loop, and this is an elementary cycle. Suppose $n > 1$. Let v be a vertex in G. If v has a self-loop, we are done. Otherwise, since v has degree 2, there must vertices v' and v'' in G, and edges (v, v') and (v, v'') in E. Let $V' = V - \{v\}$ and $E' = E \cup \{(v', v'')\} - \{(v, v'), (v,'')\}$. That is, we replace the edges from v' to v and from v to v'' with one edge that goes directly from v' to v''. Let $G' = (V', E')$. G' has $n-1$ vertices and every vertex has degree 2: Clearly every vertex other than v' and v'' has degree 2 in G', because it did in G. The vertex v' has degree 2, because all we've done is replace one of the edge (v', v) with (v', v''); similarly, v'' has degree 2. By the induction hypothesis, G' has an elementary cycle. If the cycle doesn't involve the edge (v', v''), then the cycle exists in G. If it involves (v', v''), then replace the edge (v', v'') by (v', v) and (v, v'') to get an elementary cycle in G.

[Comment: this question had nothing to do with Eulerian paths. The result actually holds for multigraphs and as long as every vertex has even degree (that is, all that matters about degree 2 is that it's even). It was not enough to say "start at any vertex and keep following edges until you get a cylce", although that was the right intuition.

7. There are $C(10, 4) = 10!/(6!4!) = 210$ 4-element subsets of A. Since the elements in each subset are between 1 and 50, the sum of the elements is at least $1+2+3+4 = 10$ and at most $47+48+49+50 = 194$. That is, there are at most $194-9 = 185$ possible sums. Now apply the pigeonhole principle, where the subsets are the pigeons and the sums are the holes. There are 210 pigeons and 185 holes. A 4-element subset B of A goes into a hole n if the sum of the sum of the elements in B is n. The pigeonhole principle says at least one of the holes has two or more pigeons, so there must be two sets whose elements sum to the same number.

8. If we assume that the teams have fixed names – i.e., Team 1, Team 2, Team 3, and Team 4 – then there are $C(12, 3)$ choices for the members of Team 1. Once the members of Team 1 have been chosen, there are 9 people left, so there are $C(9, 3)$

choices for the members of the Team 2. Similarly, there are $C(6,3)$ choices for the members of Team 3 and $C(3,3) = 1$ choice for the members of Team 4. (Once the first three teams have been chosen, the members of Team 4 are determined too.) Thus, there are $C(12,3) \times C(9,3) \times C(6,3)$ ways of splitting up the contestants into fixed teams. This is equivalent to $12!/(3! \times 3! \times 3! \times 3!)$. There's an easy way to understand why: There are 12! ways of ordering the 12 people. The first 3 go on Team 1, the next three go on Team 2, and so on. But since the first three could appear in any order without affecting Team 1, you have to divide by 3!, and similarly for Team 2, Team 3, and Team 4.

A more reasonable way of interpreting the problem is to assume that the team names don't matter. In this case, the answer is $C(12,13) \times C(9,3) \times C(6,3)/4! = 440$. [You didn't have to compute 440; it was fine to leave this answer unsimplified. We accepted the first answer too for full credit, if you made your interpretation clear; otherwise you lost .5.]

Since there are four teams, you might think that the probability of Alice and Bob being on the same team is 1/4, but this isn't quite right. Suppose there were only four people, so that each team had only one member; then surely the probability of Alice and Bob being on the same team is 0. It gets closer and closer to 1/4 the more people there are to choose from. The right way to think about this is to compute the number of ways of choosing teams where Alice and Bob are on the same team. If the teams have names, there are 4 choices for which team Alice and Bob could both be on, and 10 choices for the third member of the team. Then there are $C(9,3)$ choices for the second team, $C(6,3)$ choices for the third team, and $C(3,3) = 1$ choice for the last team. That is, there are $40 \times C(9,3) \times C(6,3)$ ways of having Alice and Bob on the same team if the team names matter. Again, we need to divide by 4! if the team names don't matter. Either way, the probability of them being on the same team is

$$\frac{40 \times C(9,3) \times C(6,3)}{C(12,3) \times C(9,3) \times C(6,3)} = \frac{2}{11}.$$

(In general, if there are $4n$ contestants that are supposed to get split into 4 teams of n players each, the probability that Alice and Bob will be on the same team is $(n-1)/(4n-1)$, so the probability really does get closer and closer to 1/4 as n grows to infinity. In this case, $n = 3$.) [Grading: you got 1 point for saying 1/4 if you gave some sort of argument for it. Just saying 1/4 without any justification was worth 0.]

9. If A and B are independent, then $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$. Notice that $A \cap B$ and $A \cap \overline{B}$ are disjoint sets, and $(A \cap B) \cup (A \cap \overline{B}) = A$. Thus, $\Pr(A \cap B) + \Pr(A \cap \overline{B}) = \Pr(A)$. It follows that $\Pr(A \cap \overline{B}) = \Pr(A) - \Pr(A \cap B) = \Pr(A) - \Pr(A)\Pr(B) = \Pr(A)(1 - \Pr(B)) = \Pr(A)\Pr(\overline{B})$. Thus, A and $\overline{B}$ are independent.

10. Let X_i be the random variable that represents the number that lands on the ith dice. The sum of the numbers that appears on the dice is thus $X_1 + X_2 + X_3$. We are interested in $E(X_1 + X_2 + X_3)$. We proved in class that $E(X_1 + X_2 + X_3) = E(X_1)+E(X_2)+E(X_3)$. Since each of $1, \ldots, 6$ is equally likely (and has probability $1/6$, $E(X_i) = (1 + \cdots + 6)/6 = 3.5$. Thus, the expected sum is $3 \times 3.5 = 10.5$.

11. (a) You need to assume that the component failures are independent. Then you can use the Bernoulli distribution.

    (b) If failures are independent, then the probability of k failures is $C(10, k)(.01)^k(.99)^{10-k}$. The probability of at least one failure is $\sum_{k=1}^{10} C(10, k)(.01)^k(.99)^{10-k}$. We gave full credit for saying this, but perhaps a better way of computing it is to observe that the probability of one or more failures is 1 minus the probability of no failures, i.e., $1 - (.99)^{10}$.

    (c) Notice that $(.99)^{10} = (1 - .01)^{10}$. By the Binomial Theorem, to two decimal places, $(1 - .01)^{10} \approx 1 - 10(.01) + C(10, 2)(.01)^2 = 1 - .1 + .0045 = .9$. Thus, the probability of one or more failures is, to two decimal places, $1 - .9 = .1$.

12. Here's the truth table for $\varphi = (\neg p \wedge (q \Rightarrow p)) \Rightarrow \neg q$:

p	q	$\neg p$	$\neg q$	$q \Rightarrow p$	$\neg p \wedge (q \Rightarrow p)$	φ
T	T	F	F	T	F	T
T	F	F	T	T	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

Since $\neg p \wedge (q \Rightarrow p)) \Rightarrow \neg q$ evaluates to T for all truth values of p and q, it is a tautology.

13. (a) $\forall x\, L(x, \text{Mary})$

    (b) $\forall x(\neg \forall y L(x, y))$ (or $\forall x \exists y \neg L(x, y)$ or $\neg(\exists x \forall y L(x, y))$).

    (c) $\exists y \forall x(\neg L(x, y)$ (or $\neg \forall y \exists x L(x, y)$).

    (d) $L(\text{Alice}, \text{Bob}) \Rightarrow L(\text{Bob}, \text{Alice})$

14. (a) The recurrence $a_{n+1} = 2a_n + (a_{n-2})^2$ has order 3 (since $n + 1 - (n - 2) = 3$). It is not linear (because of the $(a_{n-2})^2$ term) but it is homogeneous.

    (b) The characteristic equation of the recurrence is $r^2 - 2r - 3 = 0$. This factors into $(r - 3)(r + 1) = 0$, so the solutions are $r = 3$ and $r = -1$. That means the solutions to the recurrence must have the form $a_n = A3^n + B(-1)^n$. Since $a_0 = 2$, it follows that $A + B = 2$. Since $a_1 = 2$, it follows that $3A - B = 2$. Now easy algebra shows that $A = B = 1$. That means that the solution to the recurrence is $a_n = 3^n + (-1)^n$.