CS 280, Spring 1998: Final Examination

This test is out of 60; the amount for each question is as marked. Clearly explain your reasoning. (Remember, we can't read your mind! Besides, it gives us a chance to give you partial credit.) Don't forget to put your name and student number on each blue book that you use.

NOTE: You need to simplify your answers to questions 1, 4, 8, 11; don't leave factorial symbols and powers. Leave your answers to 8 and 11 as fractions in lowest terms. (The computations are easy; you don't need a calculator.)

Exams should be graded by Wednesday. I'll post a message on netnews when it's done. You can pick up your exam in 4146 Upson and get your final exam from me after the exams are graded. I'll post a message on netnews when they're available; please don't come before I post the message.

Good luck!

1. [2 points] Streets in Manhattan are laid out in a grid pattern. Suppose you're at the corner of 33rd St. and 3rd Ave. in Manhattan. You're trying to get to a restaurant on the corner of 40th St. and 7th Ave. So you have to go 7 blocks north and 4 long blocks east. How many paths are there that can get you there without walking any further than necessary?

2. [2 points] How many students must be in a class to *guarantee* that two of them have birthdays on the same day of the week?

3. [6 points] Show (a) using algebra and (b) using a combinatorial argument that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

4. [3 points] How many bit strings (strings of 0s and 1s) of length 10 have at least 8 1s in them?

5. [4 points] Prove by induction that

$$\sum_{j=n}^{2n-1} (2j+1) = 3n^2$$

if $n \geq 1$.

6. [3 points] Convert the Boolean expression $(x + y) \cdot \overline{y \cdot \overline{z}}$ to DNF.

7. [5 points] Let $L(x, y)$ stand for $x \leq y$.

   (a) Write this in English (the words x and y should *not* appear in your sentence):

   (i) $\forall x \exists y L(x, y)$

   (ii) $\exists x \forall y L(x, y)$

   (iii) $\exists y \forall x L(x, y)$

   (b) Describe a domain for which (ii) is true but (iii) is false.

   (c) Describe a domain for which (iii) is true but (ii) is false.

8. [5 points] In a certain centrally-isolated college town, fully 1/3 of the vehicles are owned by university students. A large number of the vehicles in this town are Jeeps, Broncos and the like, vehicles that are called "sports-utility vehicles" (SUVs). 35% of the students own SUVs, while 20% of the rest of the population do. If you see an SUV on the road, what is the probability that it belongs to a student? In doing this problem, clearly describe the domain and the relevant subsets (events) in that domain.

9. [2 points] What is the truth value of $(P \vee Q) \Rightarrow (P \wedge Q)$ if P and Q are false?

10. [8 points]

    (a) [4 points] Prove that $A - (B \cap C) = (A - B) \cup (A - C)$. (Note: a Venn diagram is not a proof; drawing one may help you, but it's not necessary.)

    (b) [4 points] Prove using truth tables that $P \wedge \neg(Q \wedge R)$ is logically equivalent to $(P \wedge \neg Q) \vee (P \wedge \neg R)$.

    (c) [Bonus: 0 points] What is the connection between parts (a) and (b)?

11. [10 points] Consider the following scenario. First you toss a fair coin. If it lands heads, you toss a coin that's biased towards heads, with probability 2/3 of landing heads, *twice*. (The last two coin tosses are independent.) If it lands tails, then you toss a coin that's biased towards tails, with probability 2/3 of landing tails, *once*.

    (a) [2 points] We will be interested in the expected number of tails. Carefully describe a reasonable sample space that will help us to compute it. (You don't need to calculate probabilities here; just describe the space.)

    (b) [2 points] Describe the scenario using a tree.

    (c) [3 points] Consider the events "the coin lands tails on the first coin toss" and "the coin lands tails on the second coin toss". Are these events independent? (Carefully explain why or why not; it will probably help to calculate probabilities here.)

(d) [1 point] "Number of tails" can be viewed as a random variable. Explain why.

(e) [3 points] What is the expected number of tails?

12. [4 points] Given sets S and T:

    (a) [1 point] What is $S \times T$?

    (b) [2 points] Is $S \times T = T \times S$? If so, explain why. If not, give counterexample.

    (c) [1 point] What is a *relation* on $S \times T$? (I'm looking for a definition here.)

13. [3 points]

    (a) [2 points] Are the graphs G_1 and G_2 below isomorphic? Explain why or why not.

    (b) [1 point] What is the degree of vertex b in G_1?

14. [3 points] Does the graph below have an Eulerian circuit? Explain why or why not. (I'm looking for a citation of an appropriate theorem here.) If it does, describe an Eulerian circuit in the graph.