CS280, Spring 2003: Final

1. [4 points] Prove that $A \cap (A \cup B) = A$.

   Solution: The best approach is to show that $A \cap (A \cup B) \subseteq A$ and $A \subseteq A \cap (A \cup B)$.

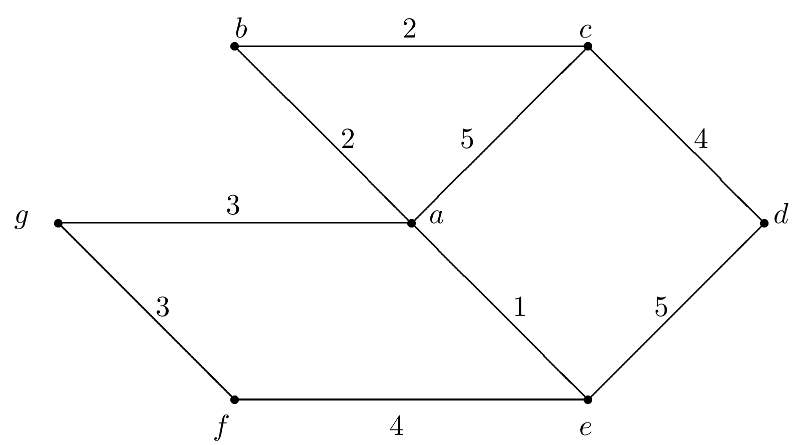
   If $x \in A \cap (A \cup B)$ then $x \in A$ (and also $x \in A \cup B$, although that's irrelevant for now), so $A \cap (A \cup B) \subseteq A$. Thus, $A \cap (A \cup B) \subseteq A$. On the other hand, if $x \in A$, then $x \in A \cup B$, so $x \in A \cap (A \cup B)$. Thus, $A \subseteq A \cap (A \cup B)$.

2. [5 points] Suppose that $a_0 = 2$ and $a_{n+1} = 2a_n + 2$. What are a_1, s_1 and a_3? Guess a formula for a_n (that does not involve a_{n-1}) and prove that your guess is correct.

   Solution: An easy computation shows that $a_1 = 6$, $a_2 = 14$, and $a_3 = 30$. [Doing this was only worth 1 point.] The fact that $a_{n+1} = 2a_n + 2$ suggests that there's doubling going on, so you should guess that a_n is roughly 2^n. Playing with it a bit, you should be able to see that $a_n = 2^{n+2} - 2$. It's easy to check that this formula works for a_0, a_1, a_2, and a_3. Now we prove that it works by induction. Let $P(n)$ be the statement that $a_n = 2^{n+2} - 2$. Clearly $P(0)$ holds. Suppose that $P(n)$ holds, so $a_n = 2^{n+2} - 2$. We want to show that $a_{n+1} = 2^{n+3} - 2$. Here's the proof:

$$\begin{aligned} a_{n+1} &= 2a_n + 2 && \text{[by definition]} \\ &= 2(2^{n+2} - 2) + 2 && \text{[induction hypothesis]} \\ &= 2^{n+3} - 4 + 2 \\ &= 2^{n+3} - 2 \end{aligned}$$

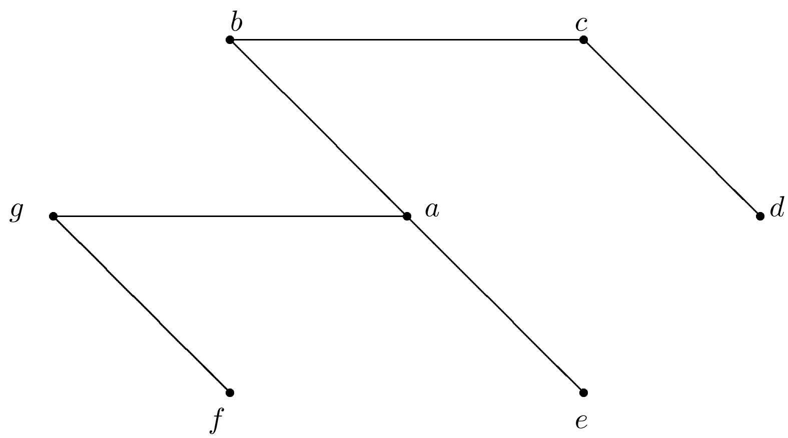
3. [4 points] Draw a minimum spanning tree for the graph below. List the order in which edges are added to the minimum spanning tree.



   Solution: The edges for the spanning tree are added in the following order:

$$(a, e), (a, b), (b, c), (a, g), (g, f), (c, d).$$

   The spanning tree looks like this:



4. [6 points] Consider the graph above for question 3, ignoring the weights.

   (a) [2 points] Does the graph have an Eulerian path? Explain why or why not. If it does have an Eulerian path, what is it?

   (b) [2 points] Does the graph have an Eulerian cycle? Explain why or why not. If it does have an Eulerian cycle, what is it?

   (c) [2 points] What is a Hamiltonian path? Does the graph have one? If so, describe it.

   Solution:

   (a) The graph has an Eulerian path, since all vertices have even degree except for c and e. One Eulerian path is
   $$c, a, e, d, c, b, a, g, f, e.$$
   Other varianta are possible, but the first and last points have to be c and e, the vertices of odd degree.

   (b) The graph does not have an Eulerian cycle, since not all vertices have even degree.

   (c) A Hamiltonian path is one which goes through all vertices exactly once. The graph has a Hamiltonian path, for example,
   $$a, b, c, d, e, f, g.$$

5. [8 points]

   (a) [3 points] How many functions are there from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$.

   (b) [3 points] How many of these functions are one-to-one?

   (c) [2 points] How many of these functions are onto?

   Solution:

   (a) There are four choices for $f(a)$, four for $f(b)$, and four for $f(c)$, so there are $4^3 = 64$ functions.

(b) For a one-to-one function, there are four choices for $f(a)$, three for $f(b)$ (since you have to choose something different from $f(a)$), and two for $f(c)$. Thus, there are $4 \times 2 = 24$ one-to-one functions.

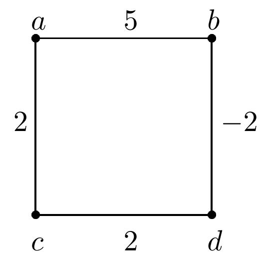
(c) There aren't any onto functions from a set of size three to one of size four.

6. [5 points] Consider the graph below.

   (a) [1 point] Just eyeballing the graph, what is the shortest path from a to d and what is its length?

   (b) [3 points] Compute the shortest path from a to d and its length using Dijkstra's algorithm. Show your work.

   (c) [1 point] Why are the answers in (a) and (b) different?



Solution:

(a) Clearly the shortest path from a to d is abd, which has length 3. (Actually, this is the shortest simple path, i.e., the shortest path with no repeated edges. There is no shortest path from a to d if repeated edges are allowed. For example, abdbd has length -1, abdbdbd has legth -5, etc. On the off chance that you noticed this – only one person did – you were given full credit.)

(b) Acording to Dijkstra's algorithm, the shortest path from a to d is acd, and it has length 4. Here's the computation:

k	$d(b)$	$d(c)$	$d(d)$	New
0	∞	∞	∞	a
1	5	2	∞	c
2	5	2	4	d

(c) Dijkstra's algorithm does not work if the edge weights are negative, as they are in this case.

7. [4 points] How many solutions (x_1, x_2, x_3, x_4) are there to the equation $x_1 + x_2 + x_3 + x_4 = 35$, where x_1, x_2, x_3, x_4 are natural numbers? (Just write down the combinatorial expression that describes the answer; there's no need to calculate it numerically.)

   Solution: This is a balls and urns problem. There are four distinguishable urns (one corresponding to each of x_1, x_2, x_3, and x_4) and 35 indistinguishable balls. From the text, the number of ways of putting b indistinguishable balls into u distinguishable urns is $C(u + b - 1, b)$. Thus, the answer is $C(38, 35)$.

8. [5 points] How many integers between 1 and 1000 are divisible by either 8 or 12?

   Solution: Let D_n be the set of numbers between 1 and 1000 divisible by n We want to compute $|D_8 \cup D_{12}$. By the inclusion-exclusion rule, $|D_8 \cup D_{12}| = |D_8| + |D_{12}| - |D_8 \cap D_{12}|$. There are $1000/8 = 125$ integers between 1 and 1000 that are divisible by 8, so $|D_8| = 125$. There are $\lfloor 1000/12 \rfloor = 83$ numbers divisible by 12, so $|D_{12} = 83|$. Note that $D_8 \cap D_{12}$ consists of those numbers divisible by both 8 and 12. Now a number is divisible by both 8 and 12 iff it's divisible by the gcd of 8 and 12, which is 24. Thus, $D_8 \cap D_{12} = D_{24}$. There are $\lfloor 1000/24 \rfloor = 41$ numbers between 1 and 1000 divisible by 24. Thus, $|D_8 \cup D_{12}| = 125+83-41 = 167$. [Grading: 1 point for realizing you had to use inclusion-exclusion, and 1 point for the calculations of D_8, 1 for the calculation of D_{12}, 1 point for realizing that $D_8 \cap D_{12}$ consists of the multiples of 24, and 1 for a reasonable calculation of $|D_8 \cap D_{12}|$, given what you thoguht it was. (For example, if you thought $D_8 \cap D_{12} = D_{96}$, and calculated D_{96} correctly, you only lost one point.)]

9. [4 points] Show that among any group of five (not necessarily consecutive) integers, there must be two whose difference is divisible by four. [Hint: think of the remainder of each of the numbers when you divide by four.]

   Solution: There are only four possible remainders when you divide by 4: 0, 1, 2, and 3. Since there are five numbers, by the pigeonhole principle, at least two of them have the same remainder r. If two numbers have the same remainder, they are both congruent to r mod 4. Thus, there difference is congruent to 0 mod 4, and thus is divisible by 4.

10. [3 points] A group of 10 women and 9 men are in a room. If five of the 19 are selected at random, what is the probability that all five are of the same sex? (Again, just write down the combinatorial expression that describes the answer; there's no need to calculate it numerically.)

    Solution: The number of ways of choosing 5 people out of town is $C(19,5)$. The number of ways of choosing 5 women is $C(10,5)$. The number of ways of choosing 5 men is $C(9,5)$. Thus, the answer is $(C(10,5) + C(9,5))/C(19,5)$. (Note that you can't use the binomial theorem here; these are not independent trials. Once you've chosen a woman, the probability of choosing another woman changes.)

11. [1 point] When we write $\Pr(X = 3)$, where X is a random variable on a sample space S, which event's probability is being calculated?

    Solution: $\Pr(X = 3) = \Pr(\{s \in S : X(s) = 3\})$.

12. [3 points] What is the probability that a biased coin, with probability 2/3 of landing heads, lands heads 4 times out of 5 flips? (Write this as a fraction in lowest terms.)

    Solution: This involves the binomial distribution. The success probabilitiy is 2/3, and you're asked for the probability of 4 successes in 5 trials. It's just

    $$C(5,4)(2/3)^4(1/3) = 5 \times 16/243 = 80/243.$$

13. [5 points] Let E be the event that a randomly generated bit string of length three (that is, a sequence of 0s and 1s of length 3) contains an odd number of 1s, and let F be the

event that the bit string starts with a 1. Are E and F independent? If so, prove it. If not, explain why.

Solution: A bit string of length 3 contains an odd number of 1s if it has either one 1 or three 1s. The probability that it has one 1 is 3/8 (this is the binomial distribution with success probability 1/2). The probability that it has three 1s is 1/8. Thus, the probability that it has an odd number of 1s in 1/2, i.e., $\Pr(E) = 1/2$. The probability that it starts with a 1 is also clearly 1/2, i.e., $\Pr(F) = 1/2$. The probability that it starts with a 1 and has an odd number of 1s is exactly the probability of getting either 100 or 111. This is clearly 1/4 (each bit string has probability 1/8. That is, $\Pr(E \cap F) = 1/4$. Since $\Pr(E \cap F) = \Pr(E) \times \Pr(F)$, it follows that E and F are indepenedent.

14. [5 points] Provide an example that shows that the variance of the sum of two random variables is not necessarily equal to the sum of their variances. More precisely, define a sample space S, a probability Pr on S, and two random variables X and Y on S such that $Var(X+Y) \neq Var(X) + Var(Y)$. [Hint: make your life (and the graders' lives) easier by taking the sample space to have two elements.]

    Solution: There was a lot of variation in this answer, since any example was acceptable. What we were looking for is (a) a clear description of the sample space S, a clear description of the probability on S, (c) a clear description of the random variables X and Y, which showed that you understood that a random variable is a function from S to the reals.) Here's one example: Let $S = \{h, t\}$. Let $\Pr(h) = \Pr(t) = 1/2$. Define $X(h) = 1$, $X(t) = 0$, and $Y(h) = 0$ and $Y(t) = 1$. Note that $X + Y(s) = X + Y(t) = 1.(Thinkofasingletossofacoin.$Xcountsthenumberoftimesthecoinlandsheads$, andsoiseither 0 or 1, dependin$
    = E(Y) = $1/2, andE(X+Y) = 1.$

    $$Var(X) = E((X - (E(X))^2) = E((X - 1/2)^2 = 1/2(0 - 1/2)^2 + 1/2(1 - 1/2)^2 = 1/4.$$

    Exactly the same reasoning shows that $Var(Y) = 1/4$. Thus $Var(X) + Var(Y) = 1/2$. But $X + Y$ is a constant function, so $Var(X+Y) = 0$. Clearly, $Var(X) + Var(Y) \neq Var(X+Y)$.

15. [4 points] Are the formulas $p \Rightarrow (q \Rightarrow r)$ and $(p \Rightarrow q) \Rightarrow r$ equivalent? (If you think they are, then show that their truth tables are identical. If you think they are not, give a truth assignment that shows they are different.)

    Solution: These formulas are not equivalent. This is probably easiest to see if we recall that $A \Rightarrow B$ is equivalent to $\neg A \vee B$. Thus, $p \Rightarrow (q \Rightarrow r)$ is equivalent to $\neg p \vee \neg q \vee r$, and $(p \Rightarrow q) \Rightarrow r$ is equivalent to $\neg(p \Rightarrow q) \vee r$, which is equivalent to $\neg(\neg p \vee q) \vee r$, which is equivalent to $(p \wedge \neg q) \vee r$. Consider the truth assignment that makes p false, q true, and r false. Under this truth assignment, $p \Rightarrow (q \Rightarrow r)$ is true, but $(p \Rightarrow q) \Rightarrow r$ is false. (You could also consider the truth assignment that makes p true, q false, and r false; it gives the same result.)

16. [6 points] Which of the following formulas is true if the domain is the natural numbers, and which are true if the domain is the real numbers. (Explain your answer in each case.)

    (a) $\exists x \exists y (x < y \wedge \forall z (z \leq x \vee y \leq z))$

(b) $\exists x \exists y (2x - y = 4 \wedge 2x + y = 6)$

(c) $\exists y \forall x ((x + y)^2 = x^2 + y^2)$

Solution:

(a) This formula says that there are two numbers x and y such that $x < y$ and there is no number between x and y. This is true in the natural numbers (take $x = 2$ and $y = 3$) and false in the reals.

(b) This formula says that there exists x and y that simultaneously solve these two equations. There are such numbers: $x = 2.5$ and $y = 1$. Thus, this formula is true in the reals and false in the natural numbers.

(c) This formula is true in both the reals and the natural numbers (take $y = 0$).

[Grading: two points for the correct answer with an explanation. You lost one point if your explanation was poor (or nonexistent).]

17. [4 points] Suppose that $T(x, y)$ means "student x takes course y". Translate each of the following sentences into first-order logic:

(a) No student takes all courses.

(b) Bob takes a course only if Ann takes the course.

(c) Every student takes some course.

(d) There is a course that no one is taking.

Solution:

(a) $\neg \exists x \forall y T(x, y)$ (or, equivalently, $\forall x \exists y \neg T(x, y)$).

(b) $\forall y T(\text{Bob}, y) \Rightarrow T(\text{Ann}, y)$

(c) $\forall s \exists y T(x, y)$.

(d) $\exists y \forall x \neg T(x, y)$.