CS280, Spring 2000: Final

This test is out of 60; the amount for each question is as marked. Clearly explain your reasoning. (Remember, we can't read your mind! Besides, it gives us a chance to give you partial credit.) Don't forget to put your name and student number on each blue book you use.

Exams should be graded by Thursday. You can pick up your exam in 4146 Upson after the exams are graded. I'll post a message on netnews and the web when they're available; please don't come before I post the message.

Good luck!

1. [3 points] Define the relation $\sim$ on $N \times N$ by taking $(m, n) \sim (k, l)$ if $m + l = n + k$. Show that $\sim$ is an equivalence relation.

2. [4 points] Inductively define $s_0 = 1$ and $s_{n+1} = 2/s_n$ for $n \in N$. Guess what s_n is and prove that your guess is correct.

3. [6 points] Here is an algorithm that factors an integer as a product of an odd integer and a power of 2.

   Input: n (a positive integer)
   $m \leftarrow n$
   $k \leftarrow 0$
   while m is even **do**
           $m \leftarrow m/2$
           $k \leftarrow k + 1$
   end

   (a) [3 points] Prove that $n = m2^k$ is a loop invariant.

   (b) [3 points] Prove that the algorithm terminates and that m is odd when it terminates.

4. [4 points]

   (a) [1 point] What is the adjacency matrix of the following multigraph (ignoring the edge weights).

   (b) [3 points] Using Dijkstra's algorithm, calculate the shortest distance from a to z. Write out the steps of the algorithm in a table as was done in class. You should stop the algorithm as soon as you have calculated the distance. Don't forget to indicate clearly what the distance is.

5. [3 points] Is it possible for an insect to crawl along the edges of a cube so as to travel along each edge exactly once? Explain why or why not.

6. [4 points] Show that every finite graph in which each vertex has degree 2 contains an elementary cycle. (Recall that an elementary cycle has no repeated vertices except for the first and last one; a self-loop counts as an elementary cycle.)

7. [5 points] Let A be a 10-element subset of $\{1, 2, 3, \ldots, 50\}$. Show that A possesses two different 4-element subsets the sum of whose elements are equal.

8. [6 points] We want to split up 12 contestants into 4 teams of 3 contestants each.

   (a) [3 points] How many ways can we do this? [Don't bother simplifying your answer.]

   (b) [3 points] What is the probability that Alice and Bob will be on the same team, if each possible way of choosing teams is equally probable. (Simplify your answer: I want an answer without factorials.)

9. [4 points] Prove, using the properties of probability, that if A and B are independent, then so are A and $\overline{B}$.

10. [3 points] What is the expected sum of the numbers that appear if three fair dice are rolled. [I am looking for an exact expression here, completely simplified. There's an easy way to do this, which requires minimal calculation.]

11. [5 points] An electronic device has 10 components. Each component has a probability .01 of failure before the warranty is up. You can calculate the probability that k components will fail using techniques we learned in class if you make one assumption.

    (a) [1 point] What is that assumption?

    (b) [2 points] Using this assumption, state the probability that one or more components will fail before the warranty is up? (No need to calculate this number exactly; just give the expression.)

    (c) [2 points] Calculate the answer in part (b) to 2 decimal places without using a calculator.

12. [4 points] Using truth tables, determine whether $(\neg p \wedge (q \Rightarrow p)) \Rightarrow \neg q$ is a tautology.

13. [4 points] Suppose $L(x, y)$ stands for "x loves y". Express the following statements in first-order logic:

    (a) Everybody loves Mary.

    (b) Nobody loves everybody.

    (c) There is somebody who nobody loves.

    (d) Alice loves Bob only if Bob loves Alice.

14. [5 points]

    (a) [3 points] What is the order of the recurrence $a_{n+1} = 2a_n + (a_{n-2})^2$? Is it linear? Is it homogeneous?

(b) [2 points] Solve the recurrence $a_n = 2a_{n-1} + 3a_{n-2}$, where $a_0 = a_1 = 2$.