Independence Example

Remember that Alice has two coins, a fair one f and a loaded one l.

- If she tosses the fair coin twice, the coin tosses are independent.
- If she tosses the biased coin twice, the coin tosses are independent.

What if she randomly picks a coin and flips it twice?

- What is the sample space?

  $\Omega = \{(f,(H,H)),(f,(H,T)),(f,(T,H)),(f,(T,T)),$
  $(l,(H,H)),(l,(H,T)),(l,(T,H)),(l,(T,T))\}.$

  The sample space has to specify which coin is picked!

- How do we construct Pr?

- E.g.: $\Pr(f,H,H)$ should be probability of getting the fair times the probability of getting heads with the fair coin: $1/2 \times 1/4$

- Follows from the following general result:

$$\Pr(A \cap B) = \Pr(B|A)\Pr(A)$$

- So with F, L denoting the events f (respectively, l) was picked,

$$\begin{aligned} \Pr\{(f,(H,H))\} &= \Pr(F \cap (H_1 \cap H_2)) \\ &= \Pr(H_1 \cap H_2|F)\Pr(F) \\ &= 1/2 \cdot 1/2 \cdot 1/2. \end{aligned}$$

  Analogously, we have for example

$$\Pr\{(l,(H,T))\} = p(1-p) \cdot 1/2.$$

Are H_1 and H_2 independent now?

Claim. $\Pr(A) = \Pr(A|E)\Pr(E) + \Pr(A|\bar{E})\Pr(\bar{E})$

Proof. $A = (A \cap E) \cup (A \cap \bar{E})$, so

$$\Pr(A) = \Pr(A \cap E) + \Pr(A \cap \bar{E}).$$

$\Pr(H_1) = \Pr(H_1|F)\Pr(F) + \Pr(H_1|L)\Pr(L) = p/2 + 1/4.$

Similarly, $\Pr(H_2) = p/2 + 1/4$.

However,

$$\begin{aligned}
& \Pr(H_1 \cap H_2) \\
= \; & \Pr(H_1 \cap H_2|F)\Pr(F) + \Pr(H_1 \cap H_2|L)\Pr(L) \\
= \; & p^2/2 + 1/4 \cdot 1/2 \\
\neq \; & (p/2 + 1/4)^2 \\
= \; & \Pr(H_1) \cdot \Pr(H_2).
\end{aligned}$$

So H_1 and H_2 are dependent events.

The Second-Child Problem

Suppose that any child is equally likely to be male or female, and that the sex of any one child is independent of the sex of the other. You have an acquaintance and you know he has two children, but you don't know their sexes. Consider the following four cases:

1. You visit the acquaintance, and a boy walks into the room. The acquaintance says "That's my older child."
2. You visit the acquaintance, and a boy walks into the room. The acquaintance says "That's one of my children."
3. The acquaintance lives in a culture, where male children are always introduced first, in descending order of age, and then females are introduced. You visit the acquaintance, who says "Let me introduce you to my children." Then he calls "John [a boy], come here!"
4. You go to a parent-teacher meeting. The principal asks everyone who has at least one son to raise their hands. Your acquaintance does so.

In each case, what is the probability that the acquaintance's second child is a boy?

- The problem is to get the right sample space

The second-ace puzzle

Alice gets two cards from a deck with four cards: A♠, 2♠, A♡, 2♡.

A♠ A♡	A♠ 2♠	A♠ 2♡
A♡ 2♠	A♡ 2♡	2♠ 2♡

Alice then tells Bob "I have an ace".

She then says "I have the ace of spades".

The situation is similar if if Alice says "I have the ace of hearts".

Puzzle: Why should finding out which particular ace it is raise the conditional probability of Alice having two aces?

The Monty Hall Puzzle

- You're on a game show and given a choice of three doors.
  - Behind one is a car; behind the others are goats.
- You pick door 1.
- Monty Hall opens door 2, which has a goat.
- He then asks you if you still want to take what's behind door 1, or to take what's behind door 3 instead.

Should you switch?

Probability Trees

Suppose that the probability of rain tomorrow is .7. If it rains, then the probability that the game will be cancelled is .8; if it doesn't rain, then the probability that it will be cancelled is .1. What is the probability that the game will be played?

The situation can be described by a tree:

Similar trees can be used to describe

- Sequential decisions
- Randomized algorithms