

Pascal's Triangle

Starting with $n = 0$, the n th row has $n + 1$ elements:

$$C(n, 0), \dots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n . This means that the LHS is *all* the ways of choosing a subset from a set of size n . The product rule says that this is 2^n .

Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\sum_{k=0}^0 \binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\sum_{k=0}^n C(n, k)$ in terms of $n - 1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

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Theorem 5:

$$(n - k) \binom{n}{k} = (k + 1) \binom{n}{k + 1} = n \binom{n - 1}{k}$$

Theorem 6:

$$\begin{aligned} C(n, k)C(n - k, j) &= C(n, j)C(n - j, k) \\ &= C(n, k + j)C(k + j, j) \end{aligned}$$

Theorem 7: $P(n, k) = nP(n - 1, k - 1)$.

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Theorem 4: For any nonnegative integer n

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned} &\sum_{k=0}^n k \binom{n}{k} \\ &= \sum_{k=1}^n k \frac{n!}{(n-k)!k!} \\ &= \sum_{k=1}^n \frac{n!}{(n-k)!(k-1)!} \\ &= n \sum_{k=1}^n \frac{(n-1)!}{(n-k)!(k-1)!} \\ &= n \sum_{k=1}^n \frac{(n-1)!}{(n-1-k)!k!} \\ &= n \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} \\ &= n \sum_{k=0}^{n-1} C(n-1, k) \\ &= n2^{n-1} \end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcommittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

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The Binomial Theorem

We want to compute $(x + y)^n$.

Some examples:

$$\begin{aligned} (x + y)^1 &= x + y \\ (x + y)^2 &= x^2 + 2xy + y^2 \\ (x + y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3 \\ (x + y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \end{aligned}$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

Proof 1: By induction on n . $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

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Inductive step:

$$\begin{aligned}
 & (x + y)^{n+1} \\
 &= (x + y)(x + y)^n \\
 &= (x + y) \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \\
 &= \sum_{k=0}^n \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^n \binom{n}{k} x^{n-k} y^{k+1} \\
 &= \dots \quad [\text{Lots of missing steps}] \\
 &= y^{n+1} + \sum_{k=0}^n \left(\binom{n}{k} + \binom{n}{k-1} \right) x^{n-k+1} y^k \\
 &= y^{n+1} + \sum_{k=0}^n \binom{n+1}{k} x^{n+1-k} y^k \\
 &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k
 \end{aligned}$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x + y)^n$?

Using the Binomial Theorem

Q: What is $(x + 2)^4$?

A:

$$\begin{aligned}
 & (x + 2)^4 \\
 &= x^4 + C(4, 1)x^3(2) + C(4, 2)x^2(2)^2 + C(4, 3)x(2)^3 + 2^4 \\
 &= x^4 + 8x^3 + 24x^2 + 32x + 16
 \end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned}
 & (1 + .02)^7 \\
 &= 1^7 + C(7, 1)1^6(.02) + C(7, 2)1^5(.0004) + C(7, 3)(.000008) + \dots \\
 &= 1 + .14 + .0084 + .00028 + \dots \\
 &\approx 1.14868 \\
 &\approx 1.1487
 \end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

In the book they talk about the *multinomial theorem*. That's for dealing with $(x + y + z)^n$.

They also talk about the *binomial series theorem*. That's for dealing with $(x + y)^\alpha$, when α is any *real* number (like 0.3).

You're not responsible for these results.