Pascal's Triangle

Starting with $n = 0$, the nth row has $n + 1$ elements:

$$C(n, 0), \ldots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\Sigma_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\Sigma_{k=0}^{n} C(n, k)$ in terms of $n - 1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

Theorem 4: For any nonnegative integer n

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned}
& \Sigma_{k=0}^{n} k\tbinom{n}{k} \\
= & \Sigma_{k=1}^{n} k\tfrac{n!}{(n-k)!k!} \\
= & \Sigma_{k=1}^{n} \tfrac{n!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=1}^{n} \tfrac{(n-1)!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=1}^{n} \tfrac{(n-1)!}{(n-k)!(k-1)!} \\
= & n\,\Sigma_{k=0}^{n-1} \tfrac{(n-1)!}{(n-1-k)!k!} \\
= & n\,\Sigma_{k=0}^{n-1} C(n-1,k) \\
= & n2^{n-1}
\end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcomittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Theorem 5:

$$(n-k)\binom{n}{k} = (k+1)\binom{n}{(k+1)} = n\binom{(n-1)}{k}$$

Theorem 6:

$$\begin{aligned} C(n,k)C(n-k,j) &= C(n,j)C(n-j,k) \\ &= C(n,k+j)C(k+j,j) \end{aligned}$$

Theorem 7: $P(n,k) = nP(n-1,k-1)$.

The Binomial Theorem

We want to compute $(x+y)^n$.
Some examples:

$$(x+y)^1 = x + y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$
\begin{aligned}
& (x+y)^{n+1} \\
= \; & (x+y)(x+y)^n \\
= \; & (x+y)\, \Sigma_{k=0}^{n} \tbinom{n}{k} x^{n-k} y^k \\
= \; & \Sigma_{k=0}^{n} \tbinom{n}{k} x^{n-k+1} y^k + \Sigma_{k=0}^{n} \tbinom{n}{k} x^{n-k} y^{k+1} \\
= \; & \ldots \qquad\qquad \text{[Lots of missing steps]} \\
= \; & y^{n+1} + \Sigma_{k=0}^{n} (\tbinom{n}{k} + \tbinom{n}{k-1})) x^{n-k+1} y^k \\
= \; & y^{n+1} + \Sigma_{k=0}^{n} (\tbinom{n+1}{k} x^{n+1-k} y^k \\
= \; & \Sigma_{k=0}^{n+1} \tbinom{n+1}{k} x^{n+1-k} y^k
\end{aligned}
$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned}
& (x+2)^4 \\
= \; & x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x 2^3 + 2^4 \\
= \; & x^4 + 8x^3 + 24x^2 + 32x + 16
\end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned}
& (1+.02)^7 \\
= \; & 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \cdot \\
= \; & 1 + .14 + .0084 + .00028 + \cdots \\
\approx \; & 1.14868 \\
\approx \; & 1.1487
\end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

In the book they talk about the *multinomial theorem.* That's for dealing with $(x + y + z)^n$.

They also talk about the *binomial series theorem.* That's for dealing with $(x+y)^\alpha$, when α is any *real* number (like 0.3).

You're not responsible for these results.