Questions/Complaints About Homework?

Here's the procedure for homework questions/complaints:

1. Read the solutions first.
2. Talk to the person who graded it (check initials)
3. If (1) and (2) don't work, talk to me.

Further comments:

- There's no statute of limitations on grade changes
  - although asking questions right away is a good strategy
- Remember that 10/12 homeworks count. Each one is roughly worth 50 points, and homework is 35% of your final grade.
  - 16 homework points = 1% on your final grade
- Remember we're grading about 80 homeworks and graders are not expected to be mind readers. It's **your** problem to write clearly.
- Don't forget to staple your homework pages together, add the cover sheet, and put your name on clearly.
  - I'll deduct 2 points if that's not the case

Strong Induction

Sometimes when you're proving $P(n+1)$, you want to be able to use $P(j)$ for $j < n$, not just $P(n)$. You can do this with *strong induction*.

1. Let $P(n)$ be the statement ... [some statement involving n]
2. The basis step
   - $P(k)$ holds because ... [where k is the base case, usually 0 or 1]
3. Inductive step
   - Assume $P(k), \ldots, P(n)$ holds. We show $P(n+1)$ holds as follows ...

Although strong induction looks stronger than induction, it's not. Anything you can do with strong induction, you can also do with regular induction, by appropriately modifying the induction hypothesis.

- If $P(n)$ is the statement you're trying to prove by strong induction, let $P'(n)$ be the statement $P(1), \ldots, P(n)$ hold. Proving $P'(n)$ by regular induction is the same as proving $P(n)$ by strong induction.

An example using strong induction

Theorem: Any item costing $n > 7$ kopecks can be bought using only 3-kopeck and 5-kopeck coins.

Proof: Using strong induction. Let $P(n)$ be the statement that n kopecks can be paid using 3-kopeck and 5-kopeck coins, for $n \geq 8$.

Basis: $P(8)$ is clearly true since $8 = 3 + 5$.

Inductive step: Assume $P(8), \ldots, P(n)$ is true. We want to show that $P(n+1)$. If $n+1$ is 9 or 10, then it's easy to see that there's no problem ($P(9)$ is true since $9 = 3 + 3 + 3$, and $P(10)$ is true since $10 = 5 + 5$). Otherwise, note that $(n+1) - 3 = n - 2 \geq 8$. Thus, $P(n-2)$ is true, using the induction hypothesis. This means we can use 3- and 5-kopeck coins to pay for something costing $n-2$ kopecks. One more 3-kopeck coin pays for something costing $n+1$ kopecks.

Binary Search

Theorem: Binary search takes at most $\lfloor \log_2(n) \rfloor + 1$ loop iterations on a list of n items.

Proof: Let $P(n)$ be the statement that if $L - F = n \geq 0$, then we go through the loop at most $\lfloor \log_2(L + 1 - F) \rfloor + 1$ times.

Basis: If $L - F = 0$, then we go through the loop at most once (0 times if the $w = w_i$ is actually on the list), and $\log_2(1) + 1 = 1$.

Inductive step: Assume $P(0), \ldots, P(n)$. If $L - F = n + 1$, then either $w = w_{\lfloor (F+L)/2 \rfloor}$ (in which case we quit), or (a) $w < w_{\lfloor (F+L)/2 \rfloor}$ or (b) $w > w_{\lfloor (F+L)/2 \rfloor}$. Let L', F' be values of L and F on the next iteration.

In case (a), $L' = \lfloor (F + L)/2 \rfloor - 1$, $F' = F$, so

$$L' + 1 - F' = \lfloor (F + L)/2 \rfloor - F = \lfloor (L - F)/2 \rfloor$$

In case (b) $F' = \lfloor (F + L)/2 \rfloor + 1$, $L' = L$, so

$$L' + 1 - F' = L - \lfloor (F + L)/2 \rfloor = \lceil (L - F)/2 \rceil$$

Either way, by strong induction, it takes at most

$$1 + \lfloor \log_2(\lceil (L - F)/2 \rceil) \rfloor + 1$$

times through the loop. (One more than it takes starting at (L', F').)

A fact about the floor function:

- $1 + \lfloor x \rfloor = \lfloor 1 + x \rfloor$ for all $x \in R$

A fact about logs:

- $1 + \log_2(x/2) = 1 + \log_2(x) - \log_2(2) = \log_2(x)$

Therefore:

$$\begin{aligned} & 1 + \lfloor \log_2(\lceil (L - F)/2 \rceil) \rfloor + 1 \\ \leq\ & 1 + \lfloor \log_2((L + 1 - F)/2) \rfloor + 1 \\ =\ & \lfloor 1 + \log_2(L + 1 - F)/2) \rfloor + 1 \\ =\ & \lfloor \log_2(L + 1 - F)) \rfloor + 1 \end{aligned}$$

This is what we wanted to prove!

Bubble Sort

Suppose we wanted to sort n items. Here's one way to do it:

Input n [number of items to be sorted]
$w_1, \ldots, w_n$ [items]

Algorithm BubbleSort
for $i = 1$ to $n - 1$
for $j = 1$ to $n - i$
if $w_j > w_{j+1}$ **then** switch(w_j, w_{j+1}) **endif**
endfor
endfor

Why is this right:

- Intuitively, because highest elements "bubble up" to the top

How many comparisons?

- Best case, worst case, average case all the same:
  - $(n-1) + (n-2) + \cdots + 1 = n(n-1)/2$

Proving Bubble Sort Correct

We want to show that the algorithm is correct by induction. What's the statement of the induction?

$P(k)$ is the statement that after k iterations of the outer loop, $w_{n-k+1}, \ldots, w_n$ are the k highest items, sorted in the right order.

Basis: How do we prove $P(1)$? By a nested induction!

This time, take $Q(l)$ to be the statement that after l iterations of the inner loop, w_{l+1} is higher than $\{w_1, \ldots, w_l\}$.

Basis: $Q(1)$ holds because after the first iteration of the inner loop, $w_2 > w_1$ (thanks to the switch statement).

Inductive step: After l iterations, the algorithm guarantees that $w_{l+1} > w_l$. Using the induction hypothesis, w_{l+1} is also higher than $\{w_1, \ldots, w_{l-1}\}$.

$Q(n-1)$ implies $P(1)$, so we're done with the base case of the main induction.

[**Note:** For a really careful proof, we need better notation (for value of w_l before and after the switch).]

Inductive step (for main induction): Assume $P(k)$. By the subinduction, after $n-k-1$ iterations of the inner loop, w_{n-k} is alphabetically after $\{w_1, \ldots, w_{n-(k+1)}\}$.

Combined with $P(k)$, this tells us $w_{n-k}, \ldots, w_n$ are the $k+1$ highest elements. This proves $P(k+1)$.

How to Guess What to Prove

Sometimes formulating $P(n)$ is straightforward; sometimes it's not. This is what to do:

- Compute the result in some specific cases
- Conjecture a generalization based on these cases
- Prove the correctness of your conjecture (by induction)

Example

Suppose $a_1 = 1$ and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$ for $n > 1$. Find an explicit formula for a_n.

Try to see the pattern:

- $a_1 = 1$
- $a_2 = a_1 + a_1 = 1 + 1 = 2$
- $a_3 = a_2 + a_1 = 2 + 1 = 3$
- $a_4 = a_2 + a_2 = 2 + 2 = 4$

Suppose we modify the example. Now $a_1 = 3$ and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$ for $n > 1$. What's the pattern?

- $a_1 = 3$
- $a_2 = a_1 + a_1 = 3 + 3 = 6$
- $a_3 = a_2 + a_1 = 6 + 3 = 9$
- $a_4 = a_2 + a_2 = 6 + 6 = 12$

$a_n = 3n!$

Theorem: If $a_1 = k$ and $a_n = a_{\lceil n/2 \rceil} + a_{\lfloor n/2 \rfloor}$ for $n > 1$, then $a_n = kn$ for $n \geq 1$.

Proof: By strong induction. Let $P(n)$ be the statement that $a_n = kn$.

Basis: $P(1)$ says that $a_1 = k$, which is true by hypothesis.

Inductive step: Assume $P(1), \ldots, P(n)$; prove $P(n+1)$.

$$\begin{aligned} a_{n+1} &= a_{\lceil (n+1)/2 \rceil} + a_{\lfloor (n+1)/2 \rfloor} \\ &= k\lceil (n+1)/2 \rceil + k\lfloor (n+1)/2 \rfloor \text{[Induction hypothesis]} \\ &= k(\lceil (n+1)/2 \rceil + \lfloor (n+1)/2 \rfloor) \\ &= k(n+1) \end{aligned}$$

We used the fact that $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n$ for all n (in particular, for $n+1$). To see this, consider two cases: n is odd and n is even.

- if n is even, $\lceil n/2 \rceil + \lfloor n/2 \rfloor = n/2 + n/2 = n$
- if n is odd, suppose $n = 2k + 1$
  - $\lceil n/2 \rceil + \lfloor n/2 \rfloor = (k+1) + k = 2k + 1 = n$

This proof has a (small) gap:

- We should check that $\lceil (n+1)/2 \rceil \leq n$

In general, there is no rule for guessing the right inductive hypothesis. However, if you have a sequence of numbers

$$r_1, r_2, r_3, \ldots$$

and want to guess a general expression, here are some guidelines for trying to find the *type* of the expression (exponential, polynomial):

- Compute $\lim_{n\to\infty} r_{n+1}/r_n$
  - if it looks like $\lim_{n\to\infty} r_{n+1}/r_n = b \notin \{0, 1\}$, then r_n probably has the form $Ab^n + \cdots$.
  - You can compute A by computing $\lim_{n\to\infty} r_n/b^n$
  - Try to compute the form of $\cdots$ by considering the sequence $r_n - Ab^n$; that is,

$$r_1 - Ab, r_2 - Ab^2, r_3 - Ab^3, \ldots$$

- $\lim_{n\to\infty} r_{n+1}/r_n = 1$, then r_n is most likely a polynomial.
- $\lim_{n\to\infty} r_{n+1}/r_n = 0$, then r_n may have the form $A/b^{f(n)}$, where $f(n)/n \to \infty$
  - $f(n)$ could be $n \log n$ or n^2, for example

Once you have guessed the form of r_n, prove that your guess is right by induction.

More examples

Come up with a simple formula for the sequence

$$1, 5, 13, 41, 121, 365, 1093, 3281, 9841, 29525$$

Compute limit of r_{n+1}/r_n:

$5/1 = 5, \quad 13/5 \approx 2.6, \quad 41/13 \approx 3.2, \quad 121/41 \approx 2.95,$
$\ldots, 29525/9841 \approx 3.000$

Guess: limit is 3 ($\Rightarrow r_n = A3^n + \cdot$)

Compute limit of $r_n/3^n$:

$1/3 \approx .33, \quad 5/9 \approx .56, \quad 13/27 \approx .5, 41/81 \approx .5,$
$\ldots, 29525/3^{10} \approx .5000$

Guess: limit is 1/2 ($\Rightarrow r_n = \frac{1}{2}3^n + \cdots) +$

Compute $r_n - 3^n/2$:

$(1 - 3/2), (5 - 9/2), (13 - 27/2), (41 - 81/2), \ldots$
$= -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \ldots$

Guess: general term is $3^n/2 + (-1)^n/2$

Verify (by induction ...)

One more example

Find a formula for

$$\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + \cdots + \frac{1}{(3n-2)(3n+1)}$$

Some values:

- $r_1 = 1/4$
- $r_2 = 1/4 + 1/28 = 8/28 = 2/7$
- $r_3 = 1/4 + 1/28 + 1/70 = (70 + 10 + 4)/280 = 84/280 = 3/10$

Conjecture: $r_n = n/(3n+1)$. Let this be $P(n)$.

Basis: $P(1)$ says that $r_1 = 1/4$.

Inductive step:

$$\begin{aligned} r_{n+1} &= r_n + \tfrac{1}{(3n+1)(3n+4)} \\ &= \tfrac{n}{3n+1} + \tfrac{1}{(3n+1)(3n+4)} \\ &= \tfrac{n(3n+4)+1}{(3n+1)(3n+4)} \\ &= \tfrac{3n^2+4n+1}{(3n+1)(3n+4)} \\ &= \tfrac{(n+1)(3n+1)}{(3n+1)(3n+4)} \\ &= \tfrac{n+1}{3n+4} \end{aligned}$$

Faulty Inductions

Part of why I want you to write out your assumptions carefully is so that you don't get led into some standard errors.

Theorem: All women are blondes.

Proof by induction: Let $P(n)$ be the statement: For any set of n women, if at least one of them is a blonde, then all of them are.

Basis: Clearly OK.

Inductive step: Assume $P(n)$. Let's prove $P(n+1)$.

Given a set W of $n+1$ women, one of which is blonde. Let A and B be two subsets of W, each of which contains the known blonde, whose union is W.

By the induction hypothesis, each of A and B consists of all blondes. Thus, so does W. This proves $P(n) \Rightarrow P(n+1)$.

Take W to be the set of women in the world, and let $n = |W|$. Since there is clearly at least one blonde in the world, it follows that all women are blonde!

Where's the bug?

Theorem: Every integer > 1 has a unique prime factorization.

[The result is true, but the following proof is not:]

Proof: By strong induction. Let $P(n)$ be the statement that n has a unique factorization, for $n > 1$.

Basis: $P(2)$ is clearly true.

Induction step: Assume $P(2), \ldots, P(n)$. We prove $P(n+1)$. If $n+1$ is prime, we are done. If not, it factors somehow. Suppose $n + 1 = rs$ $r, s > 1$. By the induction hypothesis, r has a unique factorization $\Pi_i p_i$ and s has a unique prime factorization $\Pi_j q_j$. Thus, $\Pi_i p_i \Pi_j q_j$ is a prime factorization of $n+1$, and since none of the factors of either piece can be changed, it must be unique.

What's the flaw??

Problem: Suppose $n + 1 = 36$. That is, you've proved that every number up to 36 has a unique factorization. Now you need to prove it for 36.

36 isn't prime, but $36 = 3 \times 12$. By the induction hypothesis, 12 has a unique prime factorization, say $p_1 p_2 p_3$. Thus, $36 = 3 p_1 p_2 p_3$.

However, 36 is also 4×9. By the induction hypothesis, $4 = q_1 q_2$ and $9 = r_1 r_2$. Thus, $36 = q_1 q_2 r_1 r_2$.

How do you know that $3 p_1 p_2 p_3 = q_1 q_2 r_1 r_2$.
(They do, but it doesn't follow from the induction hypothesis.)

This is a *breakdown error*. If you're trying to show something is unique, and you break it down (as we broke down $n+1$ into r and s) you have to argue that nothing changes if we break it down a different way. What if $n + 1 = tu$?

- The actual proof of this result is quite subtle

Theorem: The sum of the internal angles of a regular n-gon is $180(n-2)$ for $n \geq 3$.

Proof: By induction. Let $P(n)$ be the statement of the theorem. For $n = 3$, the result was shown in high school. Assume $P(n)$; let's prove $P(n+1)$. Given a regular $(n+1)$-gon, we can lop off one of the corners:

By induction, the sum of the internal angles of the n-gon is $180(n-2)$ degrees; the sum of the internal angles of the triangle is 180 degrees. Thus, the internal angles of the original $(n+1)$-gon is $180(n-1)$.
What's wrong??

- When you lop off a corner, you don't get a *regular* n-gon.

The fix: **Strengthen the induction hypothesis**.

- Let $P(n)$ say that the sum of the internal angles of *any* n-gon is $180(n-2)$.

Consider 0-1 sequences in which 1's may not appear consecutively, except in the rightmost two positions.

- 010110 is not allowed, but 010011 is

Prove that there are 2^n allowed sequences of length n for $n \geq 1$

Why can't this be right?

"Proof" Let $P(n)$ be the statement of the theorem.

Basis: There are 2 sequences of length 1—0 and 1—and they're both allowed.

Inductive step: Assume $P(n)$. Let's prove $P(n+1)$. Take any allowed sequence x of length n. We get a sequence of length $n+1$ by appending either a 0 or 1 at the end. In either case, it's allowed.

- If x ends with a 1, it's OK, because $x1$ is allowed to end with 2 1's.

Thus, $s_{n+1} = 2s_n = 22^n = 2^{n+1}$.

Where's the flaw?

- What if x already ends with 2 1's?

Correct expression involves separating out sequences which end in 0 and 1 (it's done in Chapter 5, but I'm not sure we'll get to it)

Inductive Definitions

Example: Define $\Sigma_{k=1}^{n} a_k$ inductively (i.e., by induction on n):

- $\Sigma_{k=1}^{1} a_k = a_1$
- $\Sigma_{k=1}^{n+1} a_k = \Sigma_{k=1}^{n} a_k + a_{n+1}$

The inductive definition avoids the use of $\cdots$, and thus is less ambiguous.

Example: An inductive definition of $n!$:

- $1! = 1$
- $(n+1)! = (n+1)n!$

Could even start with $0! = 1$.

Inductive Definitions of Sets

A *palindrome* is an expression that reads the same backwards and forwards:

- Madam I'm Adam
- Able was I ere I saw Elba

What is the set of palindromes over $\{a, b, c, d\}$? Two approaches:

1. The smallest set P such that
   (a) P contains a, b, c, d, aa, bb, cc, dd
   (b) if x is in P, then so is axa, bxb, cxc, and dxd
2. Define P_n, the palindromes of length n, inductively:
   - $P_1 = \{a, b, c, d\}$
   - $P_2 = \{aa, bb, cc, dd\}$
   - $P_{n+1} = \{axa, bxb, cxc, dxd | x \in P_{n-1}\}$, $n \geq 2$

   Let $P' = \cup_n P_n$.

Theorem: $P = P'$. (The two approaches define the same set.)
Proof: Show $P \subseteq P'$ and $P' \subseteq P$.
To see that $P \subseteq P'$, it suffices to show that

(a) P' contains a, b, c, d, aa, bb, cc, dd

(b) if x is in P', then so is axa, bxb, cxc, and dxd

(since P is the least set with these properties).

Clearly $P_1 \cup P_2$ satisfies (1), so P' does. And if $x \in P'$, then $x \in P_n$ for some n, in which case axa, bxb, cxc, and dxd are all in P_{n+2} and hence in P'. Thus, $P \subseteq P'$.

To see that $P' \subseteq P$, we prove by strong induction that $P_n \subseteq P$ for all n. Let $P(n)$ be the statement that $P_n \subseteq P$.

Basis: $P_1, P_2 \subseteq P$: Obvious.

Suppose $P_1, \ldots, P_n \subseteq P$. If $n \geq 2$, the fact that $P_{n+1} \subseteq P$ follows immediately from (b). (Actually, all we need is the fact that $P_{n-1} \subseteq P$, which follows from the (strong) induction hypothesis.)

Thus, $P' = \cup_n P_n \subseteq P$.

Recall that the set of palindromes is the smallest set P such that

(a) P contains a, b, c, d, aa, bb, cc, dd

(b) if x is in P, then so is axa, bxb, cxc, and dxd

"Smallest" is not in terms of cardinality.

- P is guaranteed to be infinite

"Smallest" is in terms of the subset relation.

Here's a set that satisfies (a) and (b) and isn't the smallest:

Define Q_n inductively:

- $Q_1 = \{a, b, c, d\}$
- $Q_2 = \{aa, bb, cc, dd, ab\}$
- $Q_{n+1} = \{axa, bxb, cxc, dxd | x \in Q_{n-1}\}$, $n \geq 2$

Let $Q = \cup_n Q_n$.

It's easy to see that Q satisfies (a) and (b), but it isn't the smallest set to do so.

Just a Reminder

(from your friendly sponsor)

What's (usually) a key step in proving a property of an algorithm:

Find a loop invariant!

- State clearly what the invariant is
- Prove that it holds (often by induction, since the invariant says "On the nth iteration of the loop, property $P(n)$ holds")