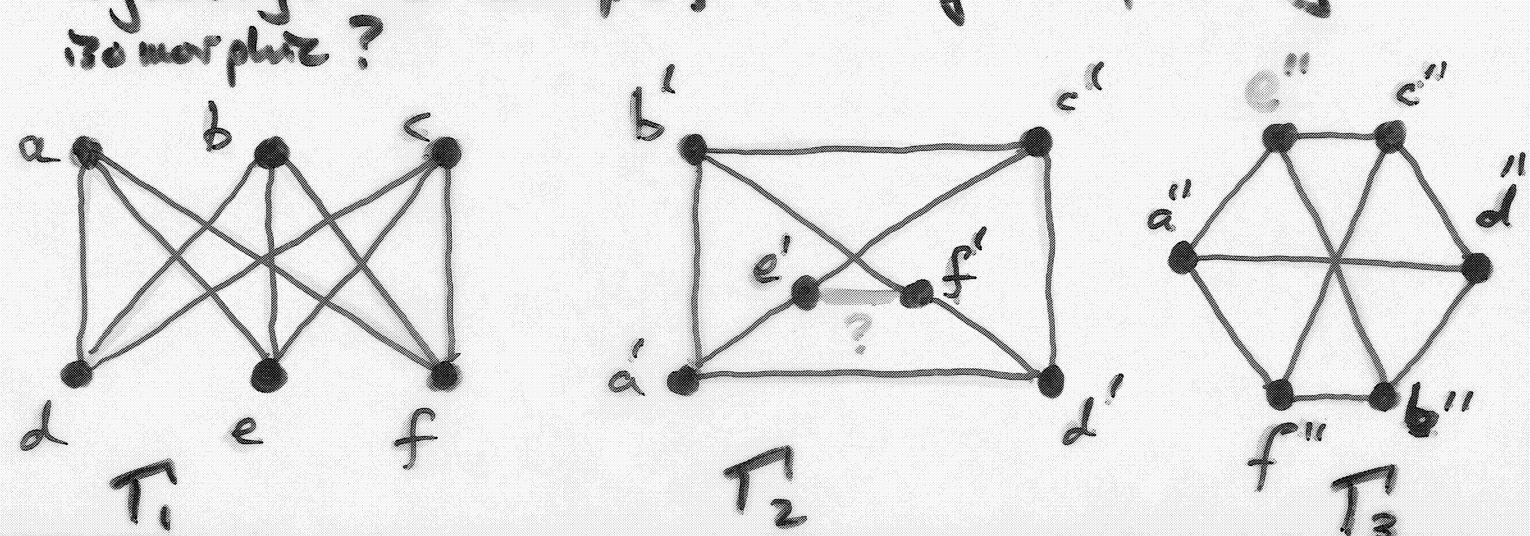
More on Graphs

We return now to look again at graphs, and will start by only considering undirected graphs. To clarify our notation; a (ν, ε)-graph is one having a total of ν nodes and ε edges, nodes joined by an edge are neighbours, the degree of a node $\deg(\nu)$ is the number of edges joining it, and the degree sequence of a graph Γ is a list of the degrees of the nodes from the largest, $\Delta(\Gamma)$, to the smallest, $\delta(\Gamma)$. There are a couple of quick observations due to Euler . . .

Fact: $\sum_{i=1}^{\nu} \deg(v_i) = 2\varepsilon$ for Γ a (ν, ε)-graph.

This follows easily since each edge has to meet two nodes. As a consequence, of course, the number of odd degree nodes in Γ must be even. We note that if every node of Γ has the same degree d, then we say that Γ is d-regular and that $\deg(\Gamma) = d$.

Two graphs $\Gamma_1 \simeq \Gamma_2$ are isomorphic if $\exists$ one to one correspondence between their sets of nodes which preserves adjacency. For example, which of the following are isomorphic?



We've talked about paths in graphs, but we can make finer distinctions:

- a *walk* is an alternating sequence of nodes and edges.
- the *length* of a walk is its number of edges.
- a *trail* is a walk whose edges are distinct.
- a *path* is a walk whose nodes are distinct ($\Rightarrow$ trail).
- a walk is *closed* if it ends where it began.
- a *cycle* is a closed path of length ≥ 3, assuming the graph has no pair of nodes joined by ≥ 2 edges.

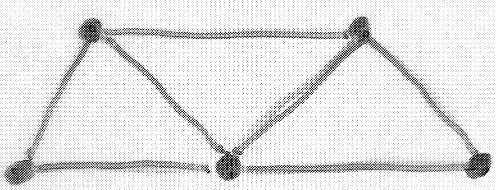
These definitions allow us to begin to give measurements of graphs:

- the *girth* of Γ, $g(\Gamma)$, is the length of its shortest cycle, and is undefined if $\nexists$ cycles.
- the *circumference* of Γ, $c(\Gamma)$, is the length of a maximal cycle, and is undefined if $\nexists$ cycles.
- a *geodesic* between v_1 and v_2 is a minimal path from v_1 to v_2, and its length is their *distance*.
- the *diameter* of Γ, $d(\Gamma)$, is the length of a maximal geodesic — Γ must be connected for this.

use to define metric on Γ

For example ...

Γ



$g(\Gamma) = 3$,
$c(\Gamma) = 5$,
$d(\Gamma) = 2$.

We define a *triangle* to be any cycle of length 3. All the graph numbers we've mentioned so far are values which are isomorphism-invariant, i.e., they are the same for graphs which are isomorphic to one another.

If Γ is a graph and v is a node of Γ, then $\Gamma - v$ is a *node-deleted subgraph* of Γ (remove also edges meeting v).

There's a famous conjecture due to Ulam and Kelly ...

Suppose Γ_1 has nodes $v_1, \ldots, v_p$
and Γ_2 has nodes $u_1, \ldots, u_p$, with $p \geq 3$.

Then $\Gamma_1 - v_i \cong \Gamma_2 - u_i \Rightarrow \Gamma_1 \cong \Gamma_2$.

This has been proven when the Γ_i are either regular graphs, disconnected graphs, or trees.

Sometimes it's useful to be able to look at the larger picture of a graph. We say that Γ is bipartite if its node set $V = V_1 \dot\cup V_2$ (disjoint union) so that each edge of Γ connects a node in V_1 with a node in V_2. It's said to be complete if it contains every edge joining V_1 and V_2, and is often denoted $K_{m,n}$ where V_1 and V_2 have m and n nodes respectively. The special case of $K_{1,n}$ is called a star.

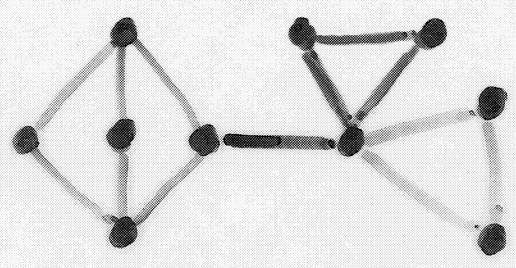
We can extend this to t-partite graphs whose nodes can be partitioned into the sets $V_1, V_2, \ldots, V_t$ in such a way that any pair of nodes are neighbours only if they live in distinct sets V_i. Completeness holds if neighbourlyness occurs if and only if nodes live in distinct V_i.

König's Theorem: Γ is bipartite iff all its cycles are even.

This follows quite simply; for if Γ has any odd cycles it cannot be bipartite; and if all cycles are even and we define V_1 and V_2 by picking a node v, putting it and all even distance nodes in V_1 with $V_2 = V - V_1$, then any edge in V_1 would create an odd cycle!

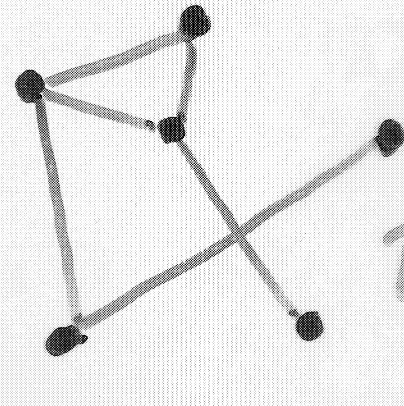
An immediate consequence of this is that Γ is bipartite iff it's 2-colourable; since we simply pick a node and colour it black, then colour all its neighbours red, continuing by alternating colours. The theorem guarantees that this will work for all cycles iff Γ is bipartite, and the non-cycles can be set to alternate until termination.

Sometimes the emphasis is more on the connectivity of a graph. We define a cutnode of Γ to be a node whose removal increases the number of connected components of Γ, and define a bridge of Γ to be an edge with the same property. Further, a non-separable graph is connected and has no cutnodes (and is non-trivial — has more than one node!). Given a graph Γ, a block of Γ is a maximal non-separable subgraph. For example...

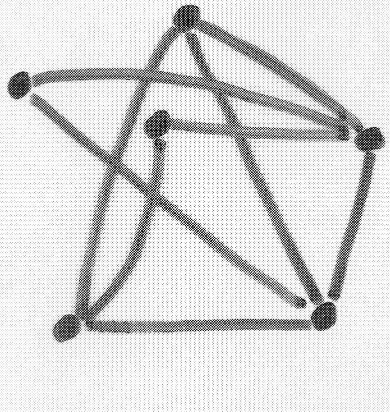


This graph has its four blocks separately coloured and has obvious cutnodes and an obvious bridge.

Given a graph Γ, its complement $\overline{\Gamma}$ has the same nodes as Γ, but a pair of nodes are joined by an edge in $\overline{\Gamma}$ iff they are not neighbours in Γ. For example...



Γ

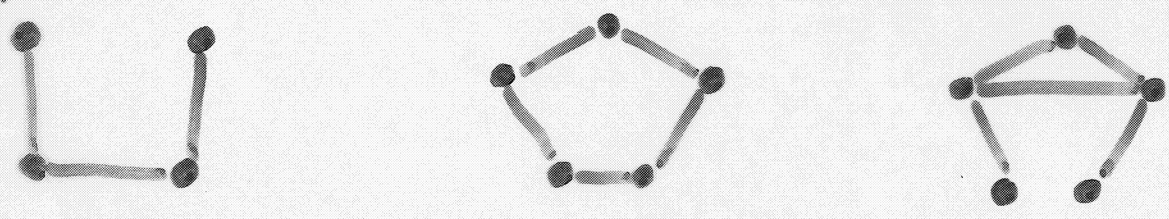


$\overline{\Gamma}$

A quick observation is that if Γ has 6 nodes (thus order 6), then either Γ or $\overline{\Gamma}$ must contain a triangle. This can be seen by picking a node v and noticing that in either Γ or $\overline{\Gamma}$, v must have at least 3 of the other 5 nodes as neighbours. Label these nodes a, b and c. If any two of a, b and c are neighbours, then that pair, together with v, form a triangle. If none of them are neighbours of one another, then they form a triangle in their complement graph!

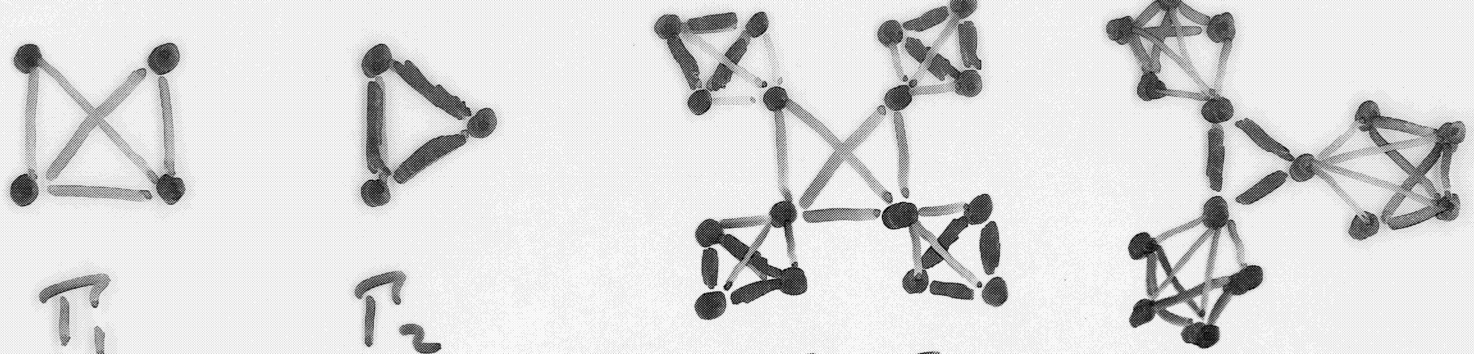
A popular interpretation of this is that any group of six folk in the same room contains either three people who know each other or three who don't. The study of the generalizations of this is called Ramsey Theory.

A graph Γ is self-complementary if $\Gamma \cong \overline{\Gamma}$. Such graphs have order $4n$ or $4n+1$ ($n \in \mathbb{Z}$). For example:



Given two graphs Γ_1 and Γ_2, we define ...

- $\Gamma_1 \cup \Gamma_2$ by having nodes $V_1 \cup V_2$ and edges $E_1 \cup E_2$.
- $\Gamma_1 + \Gamma_2$, their join, to be $\Gamma_1 \cup \Gamma_2$ with additional edges joining each node in V_1 to each node in V_2.
- $\Gamma_1 \times \Gamma_2$ has nodes $V_1 \times V_2$ with edges $(a_1, a_2) - (b_1, b_2)$ if either $a_1 = b_1$ with $a_2 - b_2 \in E_2$ or $a_2 = b_2$ with $a_1 - b_1 \in E_1$.
- $\Gamma_1 \circ \Gamma_2$, their corona, is built by pinning copies of Γ_2 to each node in Γ_1. For example...

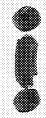
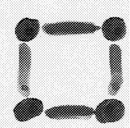
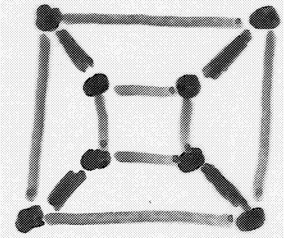


Γ_1 $\qquad$ Γ_2 $\qquad$ $\Gamma_1 \circ \Gamma_2$ $\qquad$ $\Gamma_2 \circ \Gamma_1$

Further fun constructs can be built. A graph is complete if it has all possible edges, and we denote by K_p the complete graph on p nodes. Define the 1-cube to be K_2 and then recursively, the n-cube ...

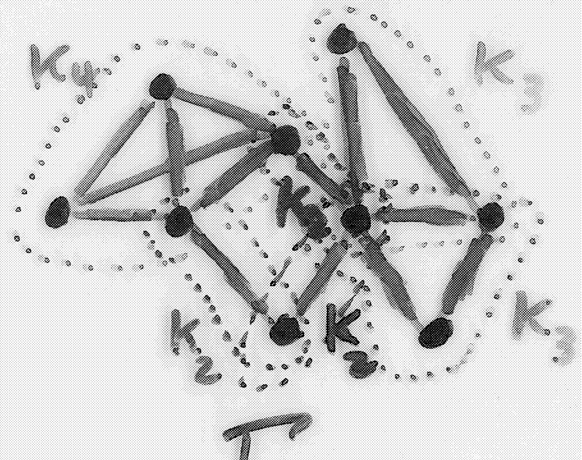
$$Q_n := K_2 \times Q_{n-1}.$$

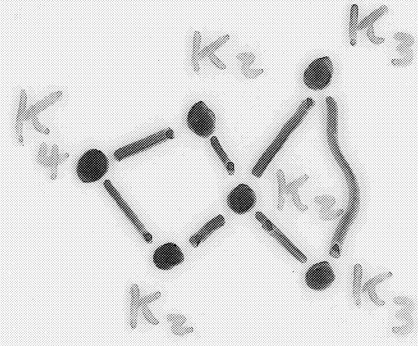
For example ...

$Q_1 =$  , $Q_2 =$  , $Q_3 =$ 

If Γ is a graph, we define its square to be Γ^2 which has the same vertices as Γ, and has an edge joining u and v iff $d(u,v) \leq 2$ in Γ. We define Γ^n in the same way, though with edges u-v iff the distance $d(u,v) \leq n$ in Γ.

Finally, in this vein, we define a clique of Γ to be a maximal complete subgraph, and we build the clique graph induced by Γ to have nodes corresponding to each clique of Γ and edges iff the corresponding cliques meet in Γ. For example ...





There are many applications where we want to find some satisfactory notion of a centre for a graph. This will often depend on the particular circumstances. For a node $v \in V_\Gamma$, we define its eccentricity by

$$e(v) := \max\{d(u,v) \mid u \in V_\Gamma\}.$$

Then the radius of Γ is

$$r(\Gamma) := \min\{e(v) \mid v \in V_\Gamma\}$$

and the diameter of Γ is

$$d(\Gamma) = \max\{e(v) \mid v \in V_\Gamma\}.$$

In this spirit, we can then define

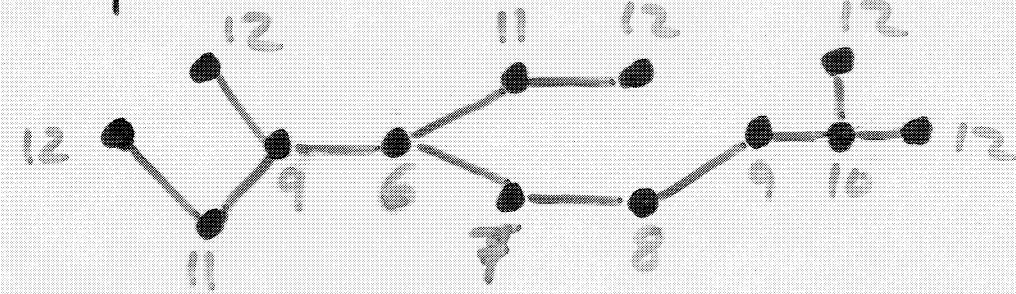
- v is a central node of Γ if $e(v) = r(\Gamma)$.
- v is a peripheral node of Γ if $e(v) = d(\Gamma)$.
- the centre of Γ is $C(\Gamma) := \{v \mid v \text{ central}\}$.
- the periphery of Γ is $\partial\Gamma := \{v \mid v \text{ peripheral}\}$.

Notice that trees will have either one central node (a central tree) or two central nodes (a bicentral tree).

To get a different kind of centre, we define a branch at node v of a tree T to be a maximal subtree for which v is an endnode. Clearly there are $\deg(v)$ branches at v. Then we define the weight at v by

$$w(v) := \max\{\#\text{edges in branches at } v\}.$$

For example ...



We've marked the weights of each of the nodes. This allows us to define the centroid of a tree to be the set of all centroid nodes, namely those nodes of minimal weight. Again, trees will have either one or two centroid nodes, and if there are two, then they will be neighbours.

This can be extended to graphs as follows. First, define for a given pair of nodes $u, v \in V_\Gamma$ the numbers

$$c_v(u) := \#\text{ nodes closer to } u \text{ than to } v$$
$$c_u(v) := \text{" " " " } v \text{ " " } u.$$

Then let $f(u,v) := c(u) - c(v)$ and

$$g(u) := \sum f(u,v) \text{ over all } v \in V_\Gamma - u.$$

The centroid of the graph Γ is then the set of all nodes $u \in \Gamma$ for which $g(u)$ is a maximum.

We say that Γ is planar if it can be drawn in the plane without crossing edges. The natural question is if there is a reasonable way to characterise planar graphs.

Define Γ_1 and Γ_2 to be homeomorphic if they can both be obtained from the same graph by sequences of edge subdivisions. Then...

Kuratowski's Theorem: Γ is planar iff it has no subgraph homeomorphic to either K_5 or $K_{3,3}$ (complete graph on 5 nodes or complete bipartite graph on two sets of 3 nodes).