- For the geometric distribution

$$M_x(t) = \ldots = \frac{p}{1-qe^t}$$

so $\mu = q/p$ and $\sigma^2 = q/p^2$.

- For the negative binomial distribution

$$M_x(t) = \ldots = p^r\left(1-qe^t\right)^{-r}$$

so $\mu = rq/p$ and $\sigma^2 = rq/p^2$.

Stochastic Processes

There's a particularly useful approach to modelling situations where things move amongst a small collection of states. Suppose we were composing a tune in C major, so using the notes...

C D E F G A B C′

We'll let the computer choose the notes according to our choice of 'transition probabilities'...

if 'in state' C go to 'state' C or E or G
with probability 0.2 each

go to state D or A or C′
with probability 0.1 each

go to state F or B
with probability 0.05 each

These transitions need to be specified for all possibilities, and of course the sum of the transition probabilities leaving each state must be 1.

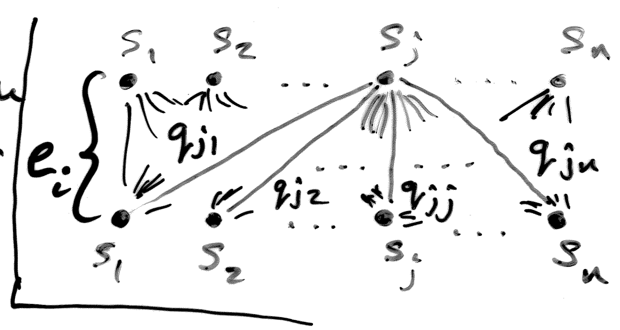
We will consider this composition as a sequence of 'events' where each event depends on at most the result of the immediately preceding event. Such a process is called a Markov Chain and is an example of a stochastic process.

We denote the 'spectrum' of probabilities that in event e_i we are in the various states s_j by a vector of probabilities $\underline{v}_i = (p_1 \; p_2 \; \cdots \; p_n)$, assuming here that there are n states. Of course, $p_1 + \cdots + p_n = 1$. We list the transition probabilities in an $n \times n$ transition matrix...

$$M = \begin{pmatrix} q_{11} & q_{21} & \cdots & q_{n1} \\ q_{12} & q_{22} & \cdots & q_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{pmatrix}$$

where $q_{ij} = \text{prob}(\text{next state} = s_j \mid \text{current state} = s_i)$.

It should be noted that authors are divided as to whether they write the matrix this way (where columns sum to 1) or the transpose of this (where rows sum to 1).

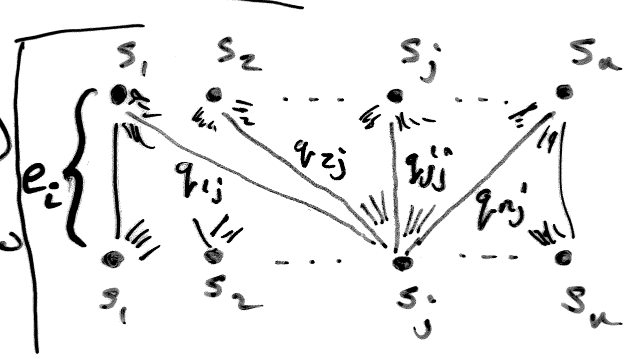


Notice that using matrix multiplication to compute the product $M\underline{v}_i$ gives a new vector whose j-th term is

$$q_{1j}p_1 + q_{2j}p_2 + \cdots + q_{nj}p_n$$

which corresponds to adding

$$p(s_j|p_1) + p(s_j|p_2) + \cdots + p(s_j|p_n)$$



so gives the probability that we're now in state s_j. Hence $M\underline{v}_i = \underline{v}_{i+1}$

So starting with an initial state vector $\underline{v}_0$ we can get the subsequent state vectors $\underline{v}_k$ by

$$\underline{v}_k = M^k \underline{v}_0$$

Indeed, we could try to find the 'steady state' condition by

$$\underline{v}_\infty = \lim_{k \to \infty} M^k \underline{v}_0 .$$

Clearly this is computationally nuts, however, if such a steady state exists, then

$$M \underline{v}_\infty = \underline{v}_\infty \quad (*)$$

so finding the steady state vector reduces to solving $(*)$. Some of you may recognise this as the eigenvector corresponding to the eigenvalue 1, which points to a far more computationally reasonable approach!

Returning to our composition, a viable transition matrix might be:

$$M = \begin{pmatrix} 0.2 & 0.1 & 0 & 0.2 & 0.3 & 0 & 0.1 & 0.4 \\ 0.1 & 0.1 & 0 & 0.1 & 0.2 & 0.3 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0 & 0.1 & 0.2 & 0 & 0.2 \\ 0.05 & 0.2 & 0.3 & 0.1 & 0.05 & 0.2 & 0 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 & 0.1 & 0 & 0.3 & 0 \\ 0.1 & 0.1 & 0.2 & 0.1 & 0.2 & 0.15 & 0 & 0 \\ 0.05 & 0.1 & 0.1 & 0 & 0.05 & 0.15 & 0.1 & 0.1 \\ 0.1 & 0 & 0 & 0.1 & 0.1 & 0 & 0.5 & 0.1 \end{pmatrix}$$

$= p(\text{next play } G \mid \text{currently playing } E)$

There's much more which can be studied on this topic!

More Counting Stuff

As we've seen, in order to compute probabilities we often need to count sizes of sets. We can facilitate this by chopping the set into blocks, and then moving these blocks around using group actions, so we'll start with a review of the relevant theory.

Definition

Let G be a group and A be a set, then a group action of G on A is a function $\varphi : G \times A \to A$, denoted $\varphi(g,a) := g \circ a$ such that

(i) $g_1 \circ (g_2 \circ a) = (g_1 g_2) \circ a \quad \forall g_1, g_2 \in G \quad \forall a \in A$

(ii) $1_G \circ a = a \quad \forall a \in A$.

Sometimes we'll choose $A = G$ and so let G 'act' on itself in some defined way.

Examples

- Let $A = \{1, 2, \ldots, n\}$ and $G = S_n$, the group of all permutations of n objects. Then we could define $g \circ a$ to be the result of applying the permutation g to a.

- Let A be a cube in $\mathbb{R}^3$ and G be the group of all rotations of A which leave A occupying the same cube-sized space in $\mathbb{R}^3$ — the "group of symmetries" of A which preserve orientation (i.e., excluding reflections). Then define $g \circ a$ as applying the rotation g to that point $a \in A$.

- Let G be any group and $A = G$ as a set. Then we could define $g \circ a := ga$ (the element of A obtained by multiplying g and a in that order). Alternatively, we could choose to define $g \circ a := gag^{-1}$ (the conjugation of a by g).

Definitions

Suppose the group G acts on the set A. Then

(i) for $a \in A$, the orbit of a is the set $\mathcal{O}(a) := \{g \cdot a \mid g \in G\}$, i.e., the set of all points of A reachable by a via G's action.

(ii) for $a \in A$, the stabilizer of a is the group $G_a := \{g \in G \mid g \cdot a = a\}$, i.e., the subgroup of those elements of G which 'fix' a.

(iii) for $g \in G$, the fixed-point set of g is the set of all points in A 'fixed' by g, notated $\mathrm{Fix}(g) := \{a \in A \mid g \cdot a = a\}$.

Remarks

Notice that $x \in \mathrm{Fix}(g)$ iff $g \in G_x$. Notice also that we can define an equivalence relation $a \sim b$ iff $b \in \mathcal{O}(a)$, so then the orbit of a is the equivalence class of a, hence A can be written as the disjoint union of orbits.

Lemma

For G acting on A we have $|\mathcal{O}(a)| = |G : G_a| \ \forall a \in A$.

Proof

Define $f : \mathcal{O}(a) \longrightarrow G/G_a$ by $f(g \cdot a) := gG_a$.

If $g \cdot a = g' \cdot a$, then $(g^{-1}g') \cdot a = a \Rightarrow g^{-1}g' \in G_a \Rightarrow gG_a = g'G_a$, so f is well-defined.

It's easy to check that f is also a bijection. //

By Lagrange's Theorem, we can see that $|G| = |\mathcal{O}(a)||G_a| \ \forall a \in A$.
If there's only one orbit, so $A = \mathcal{O}(a) \ \forall a \in A$, then we say that G acts transitively on A. Further connecting orbits and stabilizers, we have...

Lemma

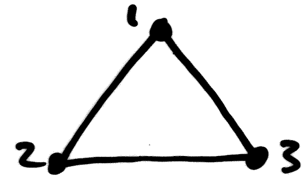
$\mathcal{O}(a) = \mathcal{O}(b) \Rightarrow a = g \cdot b$, some $g \in G \Rightarrow G_b = gG_ag^{-1}$.

Proof

Exer. — show $G_b \subseteq gG_ag^{-1}$ and $gG_ag^{-1} \subseteq G_b$.

Examples

- To get some nice geometric pictures of group actions, we'll analyse groups of <u>symmetries</u> of an object, i.e. functions which move an object so that it occupies the same spot as it did originally. For example, the symmetries of an equilateral triangle comprise



  - the identity (fixes all points)
  - two rotations r_1, r_2 of order 3 (these fix no points, r_1 rotates 120° ↺ and $r_2 = r_1^2$)
  - three reflections f_1, f_2, f_3 of order 2 (where f_i reflects through the line through the vertex i which bisects the opposite side, and hence fixes the one point i)

  If we let $A = \{1, 2, 3\}$ be the set of vertices, then

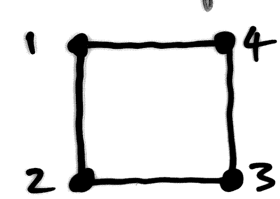
  - $\mathcal{O}(i) = A \quad \forall i \in A$, so G is transitive on A
  - $G_i = \{1, f_i\} \quad \forall i \in A$
  - $\mathrm{Fix}(1) = A, \quad \mathrm{Fix}(r_i) = \phi, \quad \mathrm{Fix}(f_i) = \{i\}$.

  If we denote the group of symmetries of this triangle by D_3, then we can generate all 6 elements of D_3 from just r_1 and f_1. In particular:

  $$1 = r_1^3 = f_1^2, \qquad f_1 = f_1,$$
  $$r_1 = r_1, \qquad f_2 = f_1 r_1, \quad \text{(apply } r_1 \text{ first then } f_1\text{)}$$
  $$r_2 = r_1^2, \qquad f_3 = f_1 r_1^2.$$

  Notice that D_3 is not Abelian, since $f_1 r_1 f_1^{-1} = r_1^{-1}$.

- Repeating the above for the group D_4 of symmetries of a square, we get



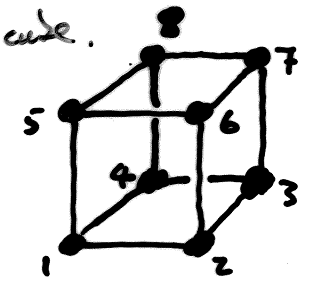
  - the identity (fixes all points)
  - 2 rotations of order 4 (no fixed points)
  - 1 " " " 2 " " "
  - 2 reflections " " 2 " " " through horiz/vertical.
  - 2 " " " 2 (2 fixed points) through diagonals

- More generally, we define the <u>dihedral group</u> D_n to be the group of isometries of a regular polygon with n sides. Then $|D_n| = 2n$, and the elements of D_n can be generated from one rotation and one reflection given that

$$r^n = 1 \,, \quad f^2 = 1 \,, \quad f r f^{-1} = r^{-1} .$$

Given that the full group of permutations S_n on n points has order $n!$ and D_n has order $2n$, we see that as n increases, D_n becomes an ever smaller subgroup of S_n.

- Let's play now in 3 dimensions. Since we're mostly concerned with concrete applications, noticing that reflection for 2-dimensional objects is equivalent to rotation in a 3^{rd} dimension leads us to recognise that actually performing a reflection of a 3-dimensional object can be a bit tricky! So we define an <u>o.p. symmetry</u> to be one which does not require reflections. (Here, "o.p." stands for orientation preserving.) Consider the o.p. symmetries of a cube.



On page 22 we saw $|G| = |O(a)|\,|G_a|$. This can be extended easily to subsets of A to give $|G| = |O(B)|\,|G_B|$ for $B \subseteq A$. For example, if B is the face $\{2,3,7,6\}$ then G_B has 4 elements (rotations) and $O(B)$ has size 6 (the six faces), so $|G| = 6 \cdot 4 = 24$. Being more detailed...

- the identity (fixes all points)
- 3 rotations of order 2 (no fixed points) around the face centres
- 3 " " " 4 (no fixed points) " " " "
- 3 " " " 4 (no fixed points) " " " "
- 6 " " " 2 (no fixed points) " " edge centres
- 8 " " " 3 (2 fixed points) " " vertices.

- As a final example before we move on to an explicit counting formula, we show that for any G of order 8 acting on a set A of size 15, there has to be at least one point fixed by all of G, i.e. $\exists\, a \in A$ with $\mathcal{O}(a) = \{a\}$ and hence $G_a = G$.
  - we know that $|\mathcal{O}(x)| \mid |G| \quad \forall x \in A$
  - if $|\mathcal{O}(x)| \neq 1$ then $|\mathcal{O}(x)|$ is even
  - but $15 = |A| = |\dot{\bigcup} \mathcal{O}(x)| = \sum |\mathcal{O}(x)|$

    remember that the $\mathcal{O}(x)$ are eq. classes.
  - hence $\exists$ at least one x for which $|\mathcal{O}(x)| = 1$.

There's a very useful counting formula due to Frobenius (but usually attributed to Burnside)...

Theorem

The number of distinct orbits is $N = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} |\mathrm{Fix}(g)|$.

Proof

Let $\sigma = \sum |\mathrm{Fix}(g)|$.

Now $x \in \mathrm{Fix}(g)$ iff $g \in G_x$, so each $x \in A$ contributes $|G_x|$ to the value of σ.

But $y \in \mathcal{O}(x) \Rightarrow |G_x| = |G_y|$, so the total contribution to σ from the points in $\mathcal{O}(x)$ is $|\mathcal{O}(x)||G_x| = |G|$.

So $\sigma = \displaystyle\sum_{g \in G} |\mathrm{Fix}(g)| = \sum_{x \in A} |G_x| = \sum_{\substack{\text{distinct}\\ \text{orbits}}} |\mathcal{O}(x)||G_x| = N|G|$. //

Examples

- Using a different colour on each face, how many different ways are there of painting a cube (using 6 colours)?
  - Two designs are the same if they differ by a rotation, so let G be the op symmetries of a cube and A be all the designs.

    Then $|G| = 24$, $|A| = 6!$ and $N = \dfrac{1}{24} \displaystyle\sum_{A} |G_x| = \dfrac{6!}{24} = 30$.

Note that the only rotation fixing any design $x \in A$ is the identity so $|G_x| = 1$ always!

- Given a circular table and six people, how many different seating arrangements are there?
  - Two seating arrangements are the same if they differ by a rotation, so with 6 people this means the corresponding group G has order 6 (namely $1, r, \ldots, r^5$ where r is rotation $60°$). Let A be all the arrangements, then $|A| = 6!$, and $|\mathrm{Fix}(1)| = 6!$, $|\mathrm{Fix}(r^i)| = 0$ for $1 \le i \le 5$, so $N = \frac{1}{6}(6! + \underbrace{0 + \ldots + 0}_{5 \text{ times}}) = 120$.

- Given 3 colours, how many different ways are there of painting the vertices of a square in 3-dimensional space?
  - Since we're in 3-dimensions, we can move the square both by rotating around and by flipping over (equivalent to the act of reflection in 2-dimensions), so two designs will be the same if they differ by an element of D_4 (page 23). Since we have 3 colours available for each vertex, the set A of all designs has size $3^4 = 81$. Recall that $|D_4| = 8$.
    Clearly $|\mathrm{Fix}(1)| = 81$.
    Let r be a rotation of order 4 and f a diagonal reflection, so the other rotations are r^2, r^3 and reflections r^2f, rf, r^3f.
    Then $|\mathrm{Fix}(r)| = |\mathrm{Fix}(r^3)| = 3$, i.e. those designs having each vertex the same colour.
    $|\mathrm{Fix}(r^2)| = 9$, i.e. those designs where diagonally opposite vertices have the same colour.
    $|\mathrm{Fix}(f)| = |\mathrm{Fix}(r^2f)| = 3^3 = 27$, since f and r^2f each fix their corresponding diagonal vertices (they reflect through that line), the colours of the non-fixed vertices must be the same, but each of the two fixed vertices can have any colour.
    $|\mathrm{Fix}(rf)| = |\mathrm{Fix}(r^3f)| = 3^2 = 9$ to match mirrored vertices.
    Hence $N = \frac{1}{8}(81 + 3 + 3 + 9 + 27 + 27 + 9 + 9) = 21$.

- How many different cubes can we make if we paint 2 faces white, 2 faces red and 2 faces blue?
  - Again let A be the set of all colourings, then $|A| = \binom{6}{2}\binom{4}{2} = 90$. The relevant group is the set of o.p. symmetries, and $|G| = 24$. $|\mathrm{Fix}(1)| = 90$. (Look again at page 24 for the group G.)

    Let r be a rotation of order 2. If it's a 'face rotation' then one pair of faces will be fixed and the other 4 swapped in pairs. Hence the swapped faces must be of the same colour, so $|\mathrm{Fix}(r)| = 3.2 = 6$. If however it's an 'edge rotation' then no faces are fixed, so all 6 are swapped in pairs. So again $|\mathrm{Fix}(r)| = 3.2 = 6$.

    Let r' be a rotation of order 3, then the orbit of any face has size 3, so we'd need to be able to paint 3 faces the same colour for a design to be fixed by r'. Hence $|\mathrm{Fix}(r')| = 0$.

    Let r'' be a rotation of order 4, then 2 faces are fixed but the other four are in an orbit of size 4, which would need to be the same colour. Hence $|\mathrm{Fix}(r'')| = 0$.

    So $N = \frac{1}{24}(90 + 9.6 + 8.0 + 6.0) = 6$.

- A flag has 6 horizontal bands (of the same width) where adjacent bands are allowed to have the same colour. If we have 3 colours to choose from for each band, how many different flags are there?
  - Let $A = \{(c_1, \ldots, c_6)\}$ be the set of all colourings. Here two flags are indistinguishable if one can be obtained from the other by reversing the order of the colours, so the relevant group $G = \{1, f\}$ where f is the reflection ($180°$ flag rotation).

    $|A| = 3^6$, so $|\mathrm{Fix}(1)| = 3^6$.

    f fixes only palindromes, so $|\mathrm{Fix}(f)| = |\text{palindromes}| = 3^3$.

    Hence $N = \frac{1}{2}(3^6 + 3^3) = 378$.