Exploiting this, there's a useful device called the <u>moment generating function</u> defined by

$$M_x(t) := E[e^{xt}],$$

which when expanded gives ...

$$M_x(t) = E\left[1 + xt + \frac{x^2t^2}{2!} + \dots + \frac{x^nt^n}{n!} + \dots\right]$$

$$= 1 + \mu t + \mu_2' \frac{t^2}{2!} + \dots + \mu_n' \frac{t^n}{n!} + \dots$$

(hence generating moments), and then we can differentiate to access the specific moments ... $\frac{d^k}{dt^k} M_x(t)\big|_{t=0} = \mu_k'$. A sometimes helpful trick is to let $R(t) := \log_e M_x(t)$, then since $M_x(0) = 1$, we get $R'(0) = M_x'(0) = \mu$ and $R''(0) = M_x''(0) - M_x'(0)^2 = \sigma^2$.

<u>Examples</u>

- Consider the Poisson distribution, so $S = \{0, 1, 2, \dots\}$ and $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$. Hence

  $$\mu = E[x] = \sum_0^\infty x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda e^{-\lambda} \sum_0^\infty \frac{\lambda^x}{x!} = \lambda.$$

  Notice that $E[e^{xt}] = \sum e^{xt} f(x)$, where the sum is taken over all $x \in S$, so here the moment generating function is

  $$M_x(t) = \sum_0^\infty e^{xt} \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_0^\infty \frac{(\lambda e^t)^x}{x!} = e^{\lambda(e^t - 1)}.$$

  Hence

  $$R(t) = \lambda(e^t - 1) \Rightarrow R^{(n)}(t) = \lambda e^t \Rightarrow \mu = \lambda \text{ and } \sigma^2 = \lambda.$$

- Consider the binomial distribution, so $S = \{0, 1, \dots, n\}$. Here the moment generating function is

  $$M_x(t) = \sum_0^n e^{xt} \binom{n}{x} p^x q^{n-x} = (pe^t + q)^n.$$

  Differentiating gives $\mu = np$ and $\sigma^2 = npq$. Notice that if $np = \lambda$ and p is small (so $q \cong 1$) then this matches the Poisson values.