Definition

Count the # events in a given continuous interval, then an approximate Poisson process with parameter $\lambda > 0$ occurs if

(i) # events in non-overlapping intervals is independent,

(ii) $p\begin{pmatrix}\text{exactly 1 event in} \\ \text{a sufficiently short} \\ \text{interval of length } h\end{pmatrix} \cong \lambda h$,

(iii) $p\begin{pmatrix}\geq 2 \text{ events in a} \\ \text{sufficiently short interval}\end{pmatrix} \cong 0$.

Now suppose we have an approx. Poisson process. Find $p(X=k)$ where k = # events in an interval of 'unit length'. To do this, partition a 'unit interval' into n chunks, then by (ii), for n sufficiently large,

$$p(\text{exactly 1 event in subinterval of length } 1/n) \cong \lambda \tfrac{1}{n},$$

and (iii) implies

$$p(\geq 2 \text{ events in } \tfrac{1}{n} \text{ subinterval}) \cong 0.$$

Now treat each $1/n$ subinterval as a Bernoulli trial, then the binomial distribution gives

$$p(X=k) \cong \binom{n}{k}(\lambda/n)^k(1-\lambda/n)^{n-k}$$

(it's approximate only because the probability $\cong \lambda/n$).

So as $n \to \infty$

$$p(X=k) \xrightarrow{\sim} \lim_{n\to\infty} \frac{n(n-1)\ldots(n-k+1)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!}\lim_{n\to\infty}\frac{n(n-1)\ldots(n-k+1)}{n^k}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= 1 \cdot \frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot 1 = \frac{\lambda^k e^{-\lambda}}{k!}$$

Hence when n is large and λ/n small, we see that the binomial distribution converges to the Poisson.