Remarks

We should observe that $(\forall x)P(x)$ represents the expression

$$P(a_1) \wedge P(a_2) \wedge \dots \wedge P(a_n) \wedge \dots$$

where the a_i range over all the values in the domain of the variable x. Similarly, $(\exists x)P(x)$ represents

$$P(a_1) \vee P(a_2) \vee \dots \vee P(a_n) \vee \dots \quad .$$

As we've remarked frequently during this course, the order matters, so in particular

$$\models (\exists y)(\forall x)P(x,y) \longrightarrow (\forall x)(\exists y)P(x,y)$$

yet the reverse direction is not valid in general; consider for example the predicate $P(x,y) \equiv$ "$x = y$" !!

We've also seen (in the first lecture) that

$$\models \neg \forall x \; P(x) \longleftrightarrow \exists x \; \neg P(x)$$

$$\models \neg \exists x \; P(x) \longleftrightarrow \forall x \; \neg P(x) .$$

Definitions

The scope of a quantifier in a formula is that formula to which the quantifier applies.

An occurence of a variable in a formula is bound iff that occurence lies within the scope of a quantifier using that variable or is the explicit occurence in a quantifier. An occurence of a variable is free if it's not bound. Furthermore, a variable itself is free in a formula iff at least one occurence is free, and it's bound iff at least one occurence is bound. Notice that a variable can be both free and bound in a formula!

Examples

bound, free

$$(\forall x)P(x,y)$$

bound, bound

$$(\exists y)(\forall x)(P(x,y) \longrightarrow (\forall z)Q(z))$$

$$(\forall z)\big(P(z) \wedge (\exists x)Q(x,z) \longrightarrow (\exists y)R(z,y)\big) \vee Q(z,x)$$

bound, bound, bound, free