- As a final example before we move on to an explicit counting formula, we show that for any G of order 8 acting on a set A of size 15, there has to be at least one point fixed by all of G, i.e. $\exists\, a \in A$ with $\mathcal{O}(a) = \{a\}$ and hence $G_a = G$.
  - we know that $|\mathcal{O}(x)| \mid |G| \quad \forall x \in A$
  - if $|\mathcal{O}(x)| \neq 1$ then $|\mathcal{O}(x)|$ is even
  - but $15 = |A| = |\dot{\bigcup}\, \mathcal{O}(x)| = \sum |\mathcal{O}(x)|$
    remember that the $\mathcal{O}(x)$ are eq. classes.
  - hence $\exists$ at least one x for which $|\mathcal{O}(x)| = 1$.

There's a very useful counting formula due to Frobenius (but usually attributed to Burnside)...

Theorem

The number of distinct orbits is $N = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$.

Proof

Let $\sigma = \sum |\mathrm{Fix}(g)|$.

Now $x \in \mathrm{Fix}(g)$ iff $g \in G_x$, so each $x \in A$ contributes $|G_x|$ to the value of σ.

But $y \in \mathcal{O}(x) \Rightarrow |G_x| = |G_y|$, so the total contribution to σ from the points in $\mathcal{O}(x)$ is $|\mathcal{O}(x)||G_x| = |G|$.

So $\sigma = \sum_{g \in G} |\mathrm{Fix}(g)| = \sum_{x \in A} |G_x| = \sum_{\substack{\text{distinct}\\\text{orbits}}} |\mathcal{O}(x)||G_x| = N|G|$. //

Examples

- Using a different colour on each face, how many different ways are there of painting a cube (using 6 colours)?
  - Two designs are the same if they differ by a rotation, so let G be the op symmetries of a cube and A be all the designs. Then $|G| = 24$, $|A| = 6!$ and $N = \frac{1}{24} \sum_{A} |G_x| = \frac{6!}{24} = 30$.

Note that the only rotation fixing any design $x \in A$ is the identity so $|G_x| = 1$ always!