CS 280 Fall '03, Prelim I Solutions

October 30, 2003

1. (a) i. Symmetric difference - $C \cap X = D$ iff $(C \cap X) + D = \emptyset$.
   ii. Use set algebra to rewrite as $(A \cap X) \cup (B \cap \overline{X}) = \emptyset$ perhaps using $X \cup \overline{X} = U$.
   iii. This is equivalent to solving $A \cap X = \emptyset$ and $B \cap \overline{X} = \emptyset$ simultaneously.
   iv. The solutions (if any) are all sets X with $B \leq X \leq \overline{A}$.

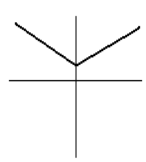
   (b) Let $x \in A \cap (B + C) \Leftrightarrow x \in A \wedge x \in B + C = (B - C) \cup (C - B) \Leftrightarrow x \in A \wedge [(x \in B \wedge x \notin C) \vee (x \in C \wedge x \notin B)] \Leftrightarrow (x \in A \wedge x \in B \wedge x \notin C) \vee (x \in A) \wedge (x \in C \wedge x \notin B) \Leftrightarrow [(x \in A \cap B) \wedge x \notin C] \vee [(x \in A \cap C) \wedge x \notin B] \Leftrightarrow x \in (A \cap B) + (A \cap C)$.

   (c) <u>B.C.</u> $x = \emptyset \Rightarrow |x| = 0$ and $P(x) = \{\emptyset\} \Rightarrow |P(x) = 1 = 2^0$ (or do $x = \{a\}$ if you prefer, though that doesn't catch the $|x| = 0$ case).

   <u>I.S.</u> $X = \{a_1, \dots, a_n, a_{n+1}$ and assume that $\forall |y| = n$, $|P(y)| = 2^n$ then $P(X) = P(\{a_1, \dots, a_n\} \cup \{a_{n+1}\}) = P(\{a_1, \dots, a_n\}) \cup \{$all subsets of X containing $a_{n+1}\} \Rightarrow |P(x)| = 2^n$ (induction hyp) $+2^n$ (since these are $\{a_{n+1}\} \cup Z$ where $Z \in P(\{a_1, \dots, a_n\})) = 2^{n+1}$.

2. (a) A relation on $X \times X$ satisfying
   i. $x \sim x \forall x \in X$,
   ii. $x \sim y \Rightarrow y \sim x \forall x_i y \in X$,
   iii. $\begin{array}{c} x \sim y \\ y \sim z \end{array} = 1x \sim z \forall xyz$.

   (b) i. $y - |x| = y - |x| \Rightarrow (x, y) \sim (x, y)$.
   ii. $(x, y) \sim (a, b) \Rightarrow y - |x| = b - |a| \Rightarrow b - |a| = y - |x| \Rightarrow (a, b) \sim (x, y)$
   iii. $\left.\begin{array}{l} (x, y) \sim (a, b) \Rightarrow y - |x| = b - |a| \\ (a, b) \sim (c, d) \Rightarrow b - |a| = d - |c| \end{array}\right\} \Rightarrow y - |x| = d - |c| \Rightarrow (x, y) \sim (c, d)$
   Graphically, $y - |x| = b - |a| \Rightarrow$ when $x = 0$ the graph cuts the y-axis at $y = b - |a|$, and it has slope $+1$ for $x > 0$ and slope -1 for $x < 0$ $(y = |x| + (b - |a|))$.



Graph of [(a,b)] Graph intersects y axis at $b - |a|$.

3. The relation $\leq$ on X will be a partial order (so making X a poset) if (i) $x \leq x \forall x \in X$, (ii) $x \leq y \wedge y \leq x \Rightarrow x = y \forall x_i y \in X$, (iii) $x \leq y \wedge y \leq z \Rightarrow x \leq z \forall x, y, z \in X$.

   (a) i. Since $xx = x \forall x \in X$, then $\underline{x} = x\underline{x} \Rightarrow x \sim x \forall x \in X$.
   ii. $x \leq y \Rightarrow x = xy$ and $y \leq x \Rightarrow y = yx = xy$ since operation commutative $= x$.
   iii. $x \leq y \Rightarrow x = xy$ and $y \leq z \Rightarrow y = yz \Rightarrow x = xy = x(yz) = (xy)z$ since operation associative $= xz \Rightarrow x \sim z$

   (b) If $\beta \in X$ satisfies $\beta \leq x \forall x \in X$, then $\beta = \beta x \forall x \in X$.

(c) Let $x, y \in X$, then $x = x^2 \Rightarrow xy = x^2y = x(xy) = (xy)x \Rightarrow (xy) \leq x$. Also $y = y^2 \Rightarrow xy = xy^2 = (xy)y \Rightarrow (xy) \leq y$.

(d) $z \leq x \Rightarrow z = zx$, and $z \leq y \Rightarrow z = zy$ so $z = zy = (zx)y = z(xy) \Rightarrow z \leq xy$.

4.

$$\begin{pmatrix} 1 & 0 & \vdots & 105 \\ 0 & 1 & \vdots & 255 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & \vdots & 105 \\ -2 & 1 & \vdots & 45 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & -2 & \vdots & 15 \\ -2 & 1 & \vdots & 45 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & -2 & \vdots & 15 \\ -17 & 5 & \vdots & 0 \end{pmatrix}$$

(a) So gcd(105,225) = 15.

(b) Also $5(105) - 2(255) = 15 \Rightarrow 25(105) - 10(255) = 75$.

(c) $6x \equiv 9 \mod 75 \ldots \gcd(6,75) = 3$, so divide by 3 to get $2x \equiv 3 \mod 25 \ldots 2^{-1} \equiv 13 \mod 25$, since $2.13 = 26 \equiv 1 \mod 25 \Rightarrow (13)2x \equiv x \equiv (13)3 \equiv 39 \equiv 14 \mod 25$. So the 3 solutions of the original equation are 14, 39, and 64.

5. (a) f is one-to-one (i.e., $f(a) = f(b) \Rightarrow a = b$) and onto (i.e., $y \in Y \Rightarrow \exists x \in X$ with $f(x) = y$).

(b) Suppose $f(X - A) = Y - f(A) \forall$ sets A. Now let $f(a) = f(b)$, then $f(X - \{a\}) = Y - f(a) = Y - f(b)$ and if $b \neq a$ then $b \in X - \{a\} \Rightarrow f(b) \in Y - f(b)\mathbb{X}$. Now notice that $Y - f(X) = f(X - X) = f(\emptyset) = \emptyset$. Now suppose f a bijection, and let A be any set. $y \in f(X - A) \Rightarrow \exists x \in X - A$ with $f(x) = y$ $x \notin A \Rightarrow y = f(x) \notin f(A) \Rightarrow y \in Y - f(A)$. $y \in Y - f(A)$ and f bijection $\Rightarrow \exists$ unique x with $y = f(x)$ $x \notin A$ (otherwise $y \notin Y - f(A)$) $\Rightarrow y = f(x) \in f(X - A)$.

(c) Let π be a permutation of $X = \{x_1, \ldots, x_n\}$. For each x_i, construct the set $[x_i] = \{x_i, \pi(x_i), \pi^2(x_i), \ldots\}$. Then X can be written as a disjoint union of $[x_{i_k}]$ for some k. Hence π can be written as a composition $\pi, \circ \ldots \circ \pi_r$ where each π_j is a cyclic permutation of one of these disjoint subsets of X. Notice that if $\pi_j = (x'_1, x'_2, \ldots x'_s)$, meaning that $\pi_j(x'_1) = x'_2 \pi_j(x'_2) = x'_3, \ldots, \pi_j(x'_s) = x'_1$, then $\pi_j = (x'_s x'_{s-1}) \circ \ldots \circ (x'_s x'_2) \circ (x'_s x'_1)$ reading the composition from left to right.

(d) Let σ_i denote a swap of a pair of elements of X, and suppose that $1 = \sigma_k \circ \ldots \circ \sigma_2 \circ \sigma_1$ where k is odd and the smallest odd number from all the odd products yielding the identity. Clearly $k \geq 3$, since $k = 1$ cannot give 1. $\sigma_k = (x_i x_j)$ for some i and j with $i < j$. From all the odd products of length k yielding 1 with $\sigma_k = (x_{ij} x_j)$, choose one having the least number of appearances of x_j in the $\sigma_1, \ldots, \sigma_n$. Since σ_k is the last swap to be performed, $\exists$ at least one σ_r with $\sigma_r = (x_i x_\alpha)$. Choose the largest such r. If x_α doesn't appear in $\sigma_{r+1}, \ldots, \sigma_{k-1}$ then σ_r commutes with all of these, so write it next to σ_k. If x_α appears in $\sigma_{r+1}, \ldots, \sigma_{k-1}$ then since $(x_\alpha x_\beta)(x_i x_\alpha) = (x_i x_\beta)(x_\alpha x_\beta)$ we can move σ_r successfully up to the $(k-1) - st$ slot, where it becomes $(x_i x_\delta)$ for some δ. If $x_\delta = x_j$ then the new σ_{k-1} and σ_k cancel $\mathbb{X}k$ min. Hence $x_\delta \neq x_j$. But then $(x_i x_j)(x_i x_\delta) = (x_j x_\delta)(x_i x_j)$ which reduces the number of appearances if x_i in the product $\mathbb{X}$.