CS 280 Fall '03, HW 7

December 1, 2003

Due in class on Friday, December 5, 2003.

1. (a) A team of 30 has 5 freshmen, 10 sophomores, 15 juniors and 5 seniors. If 8 players are selected at random, what's the probability that this sample includes exactly 2 students from each class?

   (b) You are a member of a class of 18 students. A bowl holds 18 balls, 1 blue and 17 red. Each student takes a ball from the bowl (no replacement), and the one picking the blue ball gets an "A."

      i. Your choice is to pick 1st, 5th, or last. Which is your preference?

      ii. Ditto, but now there are 2 blue balls and 16 red ones.

   (c) There are two buckets of plants. Bucket A has 24 plants which, when planted, will yield 8 yellow, 8 white, and 8 purple flowers; bucket B also has 24 plants, but these will yield 6 yellow, 6 white, and 12 purple flowers. One bucket is picked at random.

      i. If 3 plants from this bucket are planted and all give purple flowers, what's the probability that bucket B had been picked?

      ii. If, instead, 3 plants from this bucket turn out to give one of each color, what's the probability that they came from bucket A?

2. (a) Assuming that birthdays are uniformly distributed over a (non-leap) year, express the probability that n randomly selected people have dfferent birthdays. Find the smallest n for which this probability is less than 1/2.

   (b) There are k houses on a street and a postman has mail for each house. If the mail were to be perfectly shuffled and no attention given to the addresses, what's the probability that everyone receives someone else's mail?

3. (a) The <u>mode</u> of a pdf is that point in the sample space having maximum probability. Prove that the mode of a binomial distribution is $\lfloor (n+1)p \rfloor$.

   (b) Suppose your friend claims to have a biased coin, yet you doubt this. You decide to test the claim quickly by tossing the coin 5 times and only believing your friend if you get either 5 heads or 5 tails, otherwise feeling the coin to be at least close to being unbiased. Let p be the true probability of a head.

      i. if $p = 1/2$ what's the probability you'll accept the claim?

      ii. if $p = 3/4$ what's the probability you'll accept the claim?

   Sketch the graph of the probability of acceptance as a function of p.

4. A production line produces things which are deemed either OK, repairable, or useless. Assume production stable with p_1, p_2, p_3 the respective probabilities for the 3 states. If the things are packed in lots of 100,

   (a) derive an expression for the pdf of x_1, x_2, x_3, where these are the numbers of things in each lot having the respective states.

   (b) find the marginal distribution of x_1.

   (c) find the conditional distribution of x_2 given that $x_1 = 90$.

5. If the random variable $X = (x_1, x_2, x_3)$ has pdf $f(x_1, x_2, x_3)$, then x_1 is statistically independent of (x_2, x_3) if $f(x_1, x_2, x_3) = f_1(x_1) f_{23}(x_2, x_3)$ and x_1, x_2 and x_3 are statistically independent if $f(x_1, x_2, x_3) = f_1(x_1)f_2(x_2)f_3(x_3)$. Is it true that pairwise independence (x_1, x_2 independent, x_1, x_3 independent, and x_2, x_3 independent) implies x_1, x_2, and x_3 are independent? Justify your answer.

6. We define the k-th factorial moment as

$$\mu_k^* = E[x(x-1)\dots(x-k+1)]$$

   (a) Compute μ_k^* for the Poisson distribution.
   (b) Express the mean and the variance of a general discrete distribution in terms of the factorial moments.
   (c) Find the k-th factorial moments for the hypergeometric distribution, and hence its mean and variance.

7. **This is an extra credit problem, deferred from HW6.**

   Let Γ be a directed graph. Defining the directed distance in Γ from a to b as the length of the shortest path from a to b (with value ∞ if no path exists) allows us to define radius, diameter, centre, etc. for such graphs.

   Show that for Γ strongly connected with $|V| = n$, $|E| = m$,

   (a) the radius $r(\Gamma) \geq \left\lceil \frac{n-1}{m-n+1} \right\rceil$.
   (b) given the values n and m $\exists$ graph Γ for which equality holds in part (i).
   (c) if $nbr(x)$ denotes the set of all neighbors of x, and if Γ is complete, then if x_0 satisfies

$$|nbr(x_0) - \{x_0\}| = \max_{x \exists V} |nbr(x) - \{x\}|$$

   then x_0 is a centre of Γ and $r(\Gamma) \leq 2$.
   (d) if Γ is complete with $r(\Gamma) = 2$, then $\forall y \in V$ $\exists$ centre x_0 with $(x_0, y) \in E$. (A directed graph is said to be complete iff $(x, y) \notin V \Rightarrow (y, x) \in V$, i.e., iff it contains one and only one directed edge for each pair of vertices. This amounts to the underlying undirected graph being complete in the usual sense.)
   (e) if Γ is complete with $r(\Gamma) = 2$ then $\exists$ at least 3 centres.
   (f) Let Γ be strongly connected, without loops (edges from v to v), and not simply a single cycle. Show that the diameter $d(\Gamma) \geq \left\lceil \frac{2(n-1)}{m-n+1} \right\rceil$, and that given any n, m $\exists$ graph Γ for which equality holds.