CS 280 Fall '03, HW 6

November 7, 2003

Due in class on Friday, November 14, 2003.

1. For the graph in Fig. 1, evaluate the following:
   (a) the girth $g(\Gamma)$.
   (b) the circumference $c(\Gamma)$.
   (c) the diameter $d(\Gamma)$.
   (d) the minimum t for which Γ is t-partite.
   (e) draw the clique graph of Γ.
   (f) the radius of Γ.
   (g) the centre of Γ.
   (h) the periphery of Γ.
   (i) the centroid of Γ.

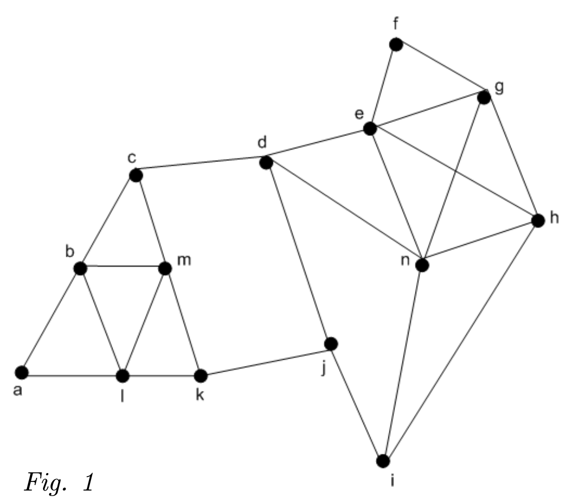

Fig. 1

2. The "sixteen puzzle" is a game involving sliding numbered slides around to reach the canonical position of Fig. 2, the questions being:

   - does a given initial configuration have a solution?
   - find an efficient solution when it exists.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Fig. 2

It's usually convenient to 'number' the empty square "16." Then the problem reduces to applying permutations to the set $\{1, 2, \ldots, 16\}$ constrained by only permitting swaps of adjacent numbers in the square. Let's call these kinds of swaps "allowable." The group of all permutations of things is denoted S_n, and we let A_n be the end group of all <u>even</u> permutations of n things, where an <u>even</u> permutation is one which can be written as a composition of an even number of swaps.

   (a) Prove that every element of A_n can be written as a composition of 3-cycles, where a 3-cycle is of the form $(\alpha\beta\gamma)$ with α, β, and γ distinct.

   (b) Let $H \subseteq S_{16}$ comprise these permutations which take the canonical configuration to a configuration using only allowable swaps. (So the effect of elements of H is to achieve a solvable configuration!).

   (c) Prove that H is a subgroup of A_{15}.

   (d) Show that $A_{15} \subseteq H$ and hence $H = A_{15}$.

   (e) Explain how to determine if a given configuration (empty square anywhere) is solvable.

3. For $d \geq 3$, suppose that Γ is a graph for which all vertices have degree $\leq d$ and suppose that K_{d+1} is <u>not</u> a subgraph of Γ. Prove that the chromatic number $\chi(\Gamma) \leq d$. (Hint: prove by contradiction and construct a minimal counterexample.)

4. (a) Let the edges of K_6 be coloured using the two colours red and blue. Show that $\exists$ at least one monochromatic triangle.

   (b) Repeat part (a) with K_7, showing $\exists$ at least four monochromatic triangles.

   (c) If now r_i denotes the number of red edges with vertex i as an endpoint and Δ denotes the number of monochromatic triangles, prove that

$$\Delta = \binom{n}{3} - \frac{1}{2}\sum_{i=1}^{n} r_i(n-1-r_i)$$

   and that

$$\Delta \geq \binom{n}{3} - \left\lfloor \frac{n}{2} \left\lfloor \left(\frac{n-1}{2}\right)^2 \right\rfloor \right\rfloor$$

5. Build a depth-first search algorithm to list the strongly connected components of a directed graph Γ.

6. The edge-connectivity of an undirected graph $\Gamma = (V, E)$ is the minimum number of edges which must be removed to disconnect the graph. Show how the edge connectivity of Γ can be determined by running a maximum flow algorithm on at most $|V|$ networks, each having $O(|V|)$ vertices and $O(|E|)$ edges.

7. Let $\Gamma = (V, E)$ be an undirected finite graph, and say that a subset $M \subseteq E$ is matching if $\forall v \in V\ \exists$ at most one $m \in M$ incident with v. Notice that there could therefore be some 'unmatched' vertices for a given M. A matching is maximal if there are no matchings on Γ having more edges.

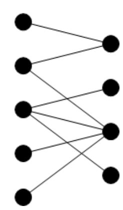

Fig. 3

   (a) Find a maximal matching for the graph in Fig. 3. How many distinct maximal matchings does this graph have?

   (b) Suppose Γ is a bipartite graph with vertex partition $V = L \cup R$, and consider a new graph $\Gamma' = (V', E')$ where $V' = V \cup \{a, b\}$ (so adding two new vertices) and

$$E' = \{(a, l)|l \in L\} \cup \{(l, r)|l \in L, r \in R, (l, r) \in E\} \cup \{(r, b)|r \in R\},$$

   with each edge of weight 1, so making Γ' into a directed graph. Show that $\exists$ matching M on Γ with $|M| = n$ iff $\exists$ flow on Γ' (from a to b) with total flow (not necessarily max) $= n$, and hence the size of a maximal matching on a bipartite graph Γ is the maximum network flow on the associated Γ'.

8. Show that for Γ strongly connected with $|V| = n$, $|E| = m$,

   (a) the radius $r(\Gamma) \equiv \left\lceil \frac{n-1}{m-n+1} \right\rceil$.

   (b) given the values n and m $\exists$ graph Γ for which equality holds in part (i).

   (c) if $nbr(x)$ denotes the set of all neighbors of x, and if Γ is a complete graph, then if n_0 satisfies

$$|nbr(x_0) - \{x_0\}| = \max |nbr(n) - \{n\}|$$

   then x_0 is a centre of $\Gamma^{x \in V}$ and $r(\Gamma) \leq 2$.

   (d) if Γ is complete with $r(\Gamma) = 2$, then $\forall y \in V\ \exists$ centre x_0 with $(x_0, y) \in E$.

   (e) if Γ is complete with $r(\Gamma) = 2$ then $\exists$ at least 3 centres.

9. Let Γ be strongly connected, without loops (edges from v to v), and not simply a single cycle. Show that the diameter $d(\Gamma) \geq \left\lceil \frac{2(n-1)}{m-n+1} \right\rceil$ with n and m as in question 6, and that given any $n, m\ \exists$ graph Γ for which equality holds.