1.A Handout 22

A. Rosen 4.6-10ac

(a) There are 6 ways to choose one croissant. By Theorem 22.5, the number of ways to choose 12 of them is $C(12 + 6 - 1, 12) = C(17, 12)$.

(c) Two dozen croissants with at least two of each kind. This corresponds to selecting 24 items with 6 different types, and there are at least two items of each type. To do this, we choose two croissants of each type (amounting to 12 chosen), then choose 12 additional croissants of any type. The number of ways to do the first choosing is exactly one. The number of ways to choose the 12 additional is $C(12 + 6 - 1, 12) = C(17, 12)$.

B. Rosen 4.6-16ad

How many solutions are there to the equation $\sum_{i=1}^{6} x_i = 29$ where

(a) all of the $x_i > 1$, that is, $x_i \geq 2$. This is exactly our second croissant problem above, but with 29 instead of 24 items to choose. This amounts to choosing $29-12$ items out of six types, i.e. $C(17+6-1, 17)$.

(d) $x_1 < 8$, $x_2 > 8$, that is, $0 \leq x_1 \leq 7$, $x_2 \geq 9$. We select from 0 to 7 items of type x_1 (call this number n_1), and nine items of type x_2. Then we select $20 - n_1$ items of types $x_2, \ldots, x_6$.

So $0 \leq n_1 \leq 7$. By the sum rule, the number is $\sum_{k=0}^{7} C(20 - k + 5 - 1, 20 - k)$.

C. Rosen 4.6-34

Putting 40 issues of a journal in four boxes, with 10 per box.

(a) How many ways to distribute them when each box is numbered? In this case we say the boxes are distinguishable, so by Theorem 22.11 on the handout, the number is $\frac{40!}{10!10!10!10!} \approx 4.705 \times 10^{21}$.

(b) When the boxes are indistinguishable, we divide the result from (a) by the number of permutations of the four boxes, i.e. the number of ways to distinguish them. (It should make sense intuitively that the number of ways to sort issues when the boxes are not marked is smaller than when they aren't.) This results in $\frac{40!}{10!10!10!10!4!} \approx 1.96 \times 10^{20}$.

D. Rosen 4.6-38

52 cards dealt to four players– how many different ways to deal the cards to four players? Each card goes in one of four distinguishable "boxes" (each box is a player), and each box has $52/4 = 13$ cards.

By Theorem 22.11 on the handout, the total number of ways to deal cards is $\frac{52!}{13!13!13!13!} \approx 5.364 \times 10^{28}$.

1.B Handout 23

A. Rosen 4.7-4

Given the set $\{1, 2, 3, 4\}$, we will denote subsets S of this set by bit strings $x_S = b_1b_2b_3b_4$, and say that $b_i = 1$ in $x_S \iff i \in S$.

$b_1b_2b_3b_4 := 0000$. `Output` $\varnothing$.
`repeat until` $b_1b_2b_3b_4 = 1111$:
  $b_1b_2b_3b_4 :=$ *next bit string*$(b_1b_2b_3b_4)$.
  `output the subset corresponding to` $b_1b_2b_3b_4$.

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end repeat
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This procedure outputs $\varnothing$, $\{4\}, \{3\}, \{3,4\}, \{2\}, \{2,4\}, \{2,3\}, \{2,3,4\}, \{1\}, \{1,4\}, \{1,3\}, \{1,3,4\}, \{1,2\}, \{1,2,4\}, \{1,2,3\}, \{1,2,3,4\}$.

B. Rosen 4.7-6

Claim: Algorithm 1 generates the next largest permutation in lexicographic order.

Proof. We will not explicitly prove all of the partial correctness assertions of the algorithm; rather, we will reason with them to get the result.

Let $a_1 \cdots a_n$ be a permutation input to Algorithm 1. The first while loop yields a j equal to the largest subscript with $a_j < a_{j+1}$. The second while loop yields k such that a_k is the smallest integer greater than a_j that is $\geq j$. The algorithm interchanges a_k and a_j, then reverses the order of those elements after a_j. The algorithm thus returns a permutation of the form $a_1 \cdots a_{j-1} a_k a_n \cdots a_{j+1}$, where a_k is removed from the second $\cdots$ (assuming $k \neq n$ and $k \neq j+1$).

It should be clear that the above results in a permutation which is lexicographically larger than the input. It remains to show that it is the smallest permutation which is larger. The next largest permutation $a_{i_1} \cdots a_{i_n}$ after $a_1 \cdots a_n$ can only differ in those positions $j, j+1, \ldots, n$. If it differed in a position $m < j$, that would result in a permutation between $a_{i_1} \cdots a_{i_n}$ and $a_1 \cdots a_n$ in lexicographic order (namely, the permutation output by the algorithm).

a_{i_j} should thus have the next largest element out of $a_{j+1}, \ldots, a_n$. This is precisely a_k in the algorithm. Finally, since j was the largest position in which $a_j < a_{j+1}$, for all positions m greater than j, $a_m \geq a_{m+1}$. Thus, reversing the order of all the elements after position j enforces that $a_m \leq a_{m+1}$; i.e. the elements occur in increasing order.

Any permutation greater than $a_1 \cdots a_n$ is greater than or equal to this, since it must have $a_1 \cdots a_{j-1} a_k$ in its first through jth positions, followed by some permutation of $a_j, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n$. The ordering we have chosen (increasing order) yields the smallest such permutation.

Hence our permutation is the next largest.

□

1.C Handout 24

A. Rosen 5.1-10

(a)

$a_0 = 50,000$

$a_n = (105/100)a_{n-1} + 1000 = 21/20a_{n-1} + 1000$.

(b)

In 1995, $n = 8$.

$a_8 = (21/20)a_7 + 1000 = (21/20)[(21/20)a_6 + 1000] + 1000$

$= (21/20)[(21/20)a_6 + 1000] + 1000$

$= (21/20)[(21/20)[(21/20)[(21/20)[(21/20)[(21/20)[(21/20)[(21/20) \cdot 50000 + 1000] + 1000] + 1000] + 1000] + 1000] + 1000] + 1000] + 1000.$

≈ 83421.88.

(c)

An explicit formula: $a_0 = 50000$, $a_n = (21/20)^n \cdot 50000 + \sum_{k=0}^{n-1}(21/20)^k \cdot 1000$.

B. Rosen 5.1-22

(a)

Recurrence for number of ways to climb n stairs, given that a person can climb either one, two, or three stairs at a time:

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

(To get to n steps, the last move made by the person is either one, two, or three stairs.)

(b)

Initial conditions are $a_1 = 1$, $a_2 = 2$, $a_3 = 4$.

(c)

$a_4 = 4 + 2 + 1 = 7$, $a_5 = a_4 + a_3 + a_2 = 7 + 4 + 2 = 13$, $a_6 = 13 + 7 + 4 = 24$, $a_7 = 24 + 13 + 7 = 44$, $a_8 = 44 + 24 + 13 = 81$.

C. Rosen 5.1-30

(a)

Using nickels and dimes to pay for n cents, where order matters.

We assume nickels and dimes are indistinguishable, so the problem amounts to counting permutations of sums of 5 and 10 which add up to n.

First, if n is not divisible by 5, then there are 0 ways to pay for n cents using just dimes and nickels. So we will assume n is a multiple of 5.

We use a recurrence relation:

$a_n = a_{n-5} + a_{n-10}$.

Consider the "last" coin paid in our sequence of coins used to obtain n cents. It is either a nickel (in which case we consider each list of coins before it that add up to $n - 5$ cents), or it is a dime ($n - 10$ cents).

Initial conditions are $a_5 = 1$, $a_{10} = 2$.

For an example, note that $a_{15} = 3$. (The sequences are *nickel, nickel, nickel*, *nickel, dime*, and *dime, nickel*.)

(b)

How many ways to pay 45 cents?

From above, we have $a_5 = 1$, $a_{10} = 2$, $a_{15} = 3$.

$a_{20} = 5$, $a_{25} = 8$, $a_{30} = 13$, $a_{35} = 21$, $a_{40} = 34$, $a_{45} = 55$.