1.A Handout 11

11A. Section 2.6, 2b

$$A+B=\begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix}+\begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix}=\begin{bmatrix} -4 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix}$$

11B. Section 2.6, 4c

$$AB=\begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} (0\cdot 4+-1\cdot -2) & (0\cdot -1+-1\cdot 0) & (0\cdot 2+-1\cdot 3) & (0\cdot 3+-1\cdot 4) & (0\cdot 0+-1\cdot 1) \\ (7\cdot 4+2\cdot -2) & (7\cdot -1+2\cdot 0) & (7\cdot 2+2\cdot 3) & (7\cdot 3+2\cdot 4) & (7\cdot 0+2\cdot 1) \\ (-4\cdot 4+-3\cdot -2) & (-4\cdot -1+-3\cdot 0) & (-4\cdot 2+-3\cdot 3) & (-4\cdot 3+-3\cdot 4) & (-4\cdot 0+-3\cdot 1) \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 0 & -3 & -4 & -1 \\ 24 & -7 & 20 & 29 & 2 \\ -10 & 4 & -17 & -24 & -3 \end{bmatrix}$$

11C. Section 2.6, 24a

A_1 is 20×50, A_2 is 50×10, A_3 is 10×40.

We examine the two possible cases. We will count only multiplications as they are more significant operations than addition (and this is the way the book makes these quantitative comparisons).

$(A_1A_2)A_3$: Using the standard algorithm, $20\cdot 50\cdot 10=10000$ multiplications are done for computing (A_1A_2). Since this resulting matrix is 20×10, the multiplication of it with A_3 uses $20\times 10\times 40=8000$ multiplications. Hence 18000 multiplications in all.

$A_1(A_2A_3)$: $50\cdot 10\cdot 40=20000$ multiplications are done for computing (A_2A_3). Thus computing the product in this case is more expensive than the first case.

11D. Section 2.6, 28

(a) $A\vee B=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) $A\wedge B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(c) $A\odot B=\begin{bmatrix} (1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 0) \\ (0\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(1\wedge 0) \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

1.B Handout 12

12A. Section 3.1, 2ade

(a) Simplification is used here.
(d) Addition is used here.
(e) Hypothetical syllogism used.

12B. Section 3.1, 10acd

(a) Let $P(x)$ = "x owns a red convertible", and $Q(x)$ = "x has gotten a speeding ticket".
Then we are asserting $P(Linda)$, $\forall x P(x) \rightarrow Q(x)$, where the domain of x is the set of students in the class. From these we may assert $P(Linda) \rightarrow Q(Linda)$ by universal instantiation, then $Q(Linda)$ by modus ponens, then $\exists x Q(x)$ by existential generalization.

(c) Let $P(x)$ = "x is produced by John Sayles", $Q(x)$ = "x is wonderful", and $R(x)$ = "x is about coal miners".
Then our assertions are that $\forall x P(x) \rightarrow Q(x)$, and $\exists x P(x) \wedge R(x)$, where the domain of x is the set of movies. Then by existential instantiation, $P(c) \wedge R(c)$ for some movie c. By simplification, $P(c)$. By universal instantiation, $P(c) \rightarrow Q(c)$. By modus tollens, $Q(c)$. Since $P(c)$, $R(c)$ and $Q(c)$, by conjunction $P(c) \wedge R(c) \wedge Q(c)$. Finally by existential generalization, $\exists x P(x) \wedge R(x) \wedge Q(x)$.

(d) Let $P(x)$ = "x has been to France", $Q(x)$ = "x has visited the Louvre".
Then we assert the propositions p_1 = "$\exists x P(x)$", and q = "$\forall y P(y) \rightarrow Q(y)$", where x is quantified over the domain of students in the class, and y is quantified over the domain of all people. Since the set of students is a subset of the set of people, we are implicitly assuming that $\forall x P(x) \rightarrow Q(x)$ is true as well.
By existential instantiation on p_1, $P(c)$ for some student c. By universal instantiation on q, $P(c) \rightarrow Q(c)$, since the student c is in the domain of people. Therefore by modus ponens, $Q(c)$. By existential generalization, $\exists x Q(x)$.

12C. Section 3.1, 12
The flaw is in the step "$n^2 \neq 3k$ for some integer k implies $n \neq 3l$ for some integer l." The reasoning is circular since this statement is equivalent to what we are trying to prove, and no justification for this statement is provided.

12D. Section 3.1, 26
Claim: There is an integer n such that $2^n + 1$ is not prime.
Consider $n = 5$, so $2^5 + 1 = 33$. Clearly, $33 = 11 \cdot 3$, so the claim is true for $n = 5$.

1.C Handout 13

13A. Section 3.2, 2
The sum of the first n even positive integers can be expressed using the following formal notation: $\sum_{k=1}^{n} 2k$. [By convention, the "empty sum" $\sum_{k=1}^{0} 2k$ is 0.]
Formally then, our **claim** is: $P(n)$ holds for all natural numbers n, where $P(n)$ is the statement $\sum_{k=1}^{n} 2k = n(n+1)$.
Proof by induction on n, with $P(n)$ as the induction hypothesis. Base case is $P(0)$. The sum is 0 and $0 \cdot (0+1) = 0$.
Induction step. Assume $P(n)$ is true. In the case of $P(n+1)$:
$\sum_{k=1}^{n+1} 2k = \sum_{k=1}^{n} 2k + 2(n+1)$
$= n(n+1) + 2(n+1)$ by induction hypothesis
$= (n+1)(n+2)$. By induction, $P(n)$ holds for all natural numbers n.

□

13B. Section 3.2, 14

Claim: For any integer $n > 1$, $n! < n^n$.
Proof. By induction on n. [The induction hypothesis is $n! < n^n$.] Base case is $n = 2$; in this case $2 = 2! < 2^2 = 4$.

Induction step: Assume the claim is true for n. Then $(n+1)! = (n+1)n! < (n+1)n^n$ by induction hypothesis. Furthermore, $(n+1)n^n < (n+1)(n+1)^n = (n+1)^{n+1}$, since $n > 1$. The claim holds for $n+1$, therefore by induction the claim holds in general.

$\square$

13C. Section 3.2, 20

Claim: For any integer $n \geq 0$, 3 divides $n^3 + 2n$.
Proof. By induction on n. [The induction hypothesis is 3 divides $n^3 + 2n$.] Base case is when $n = 0$, and 3 divides $0^3 + 2 \cdot 0 = 0$ trivially.

Induction step: Assume claim is true for n. We must check to see if 3 divides $(n+1)^3 + 2(n+1)$.

$(n+1)^3 + 2(n+1) = (n^3 + 2n) + 3n^2 + 3n + 3$. By induction hypothesis, there exists a k such that $3k = n^3 + 2n$. Therefore $(n+1)^3 + 2(n+1) = 3k + 3n^2 + 3n + 3 = 3(k + n^2 + n + 1)$, and 3 divides $(n+1)^3 + 2(n+1)$. So the claim holds for $n+1$.

$\square$

13D. Section 3.2, 48

The high-level structure of the proof is legitimate, formally speaking. (Recall the second principle of mathematical induction.) The low-level reasoning in the body of the inductive step is where the logical flaw lies.

Specifically, he (tacitly) infers the equation $a^{n-1} = 1$ from the hypothesis $\forall k[0 \leq k \leq n \rightarrow a^k = 1]$, a step that is valid only if $0 \leq n-1 \leq n$. Although the $n-1 \leq n$ part of that implicit assumption can easily be justified, the $0 \leq n-1$ part is unwarranted. Indeed, when $n = 0$, i.e., when we're "proving the $P(1)$ case," the quantity $n-1$ is negative.