CS280, Spring 2001	Handout 6
February 2, 2001	

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 1.8

2. The main message of this lecture:

> **There are two good reasons to compare functions according to their asymptotic behavior. First: very often it takes big values of the argument for a function to show its real strength. Second: often (but not always!) big arguments correspond to real size problems.**

Real numbers (as well as rationals, integers, natural numbers) are linearly ordered: for any two numbers a, b either $a < b$ or $a > b$ or $a = b$. Unfortunately, this does not hold for functions. Suppose by $f \leq g$ we understand the condition $\forall x(f(x) \leq g(x))$, for example, $\cos x \leq x^2 + 1$ since for all $x \in \mathbf{R}$ $\cos x \leq 1$ and $1 \leq x^2 + 1$. Then too many pairs of functions become incomparable. for example, $\cos x$ versus x^2. For some of the x's, e.g. $x = 0$ $\cos x$ is greater then the squaring function: $\cos 0 = 1 > 0 = 0^2$. However, for $x \geq 1$ the latter clearly dominates: $1^2 = 1$, $2^2 = 4$, $\ldots, 10^2 = 100, \ldots$, whereas $\cos x$ stays somewhere between -1 and 1. Another typical example: x^2 versus $x + 100$. For small x's the squaring function again loses: $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $\ldots$ whereas the linear function with a "big" constant $x + 100$ scores much better points: $0 + 100 = 100$, $1 + 100 = 101$, $2 + 100 = 102$, $\ldots$. However, the rate of growth of x^2 is much higher and this function catches up quickly: $10^2 = 100 < 10 + 100 = 110$, but $11^2 = 121 > 11 + 100 = 111$, $\ldots$ $20^2 = 400 > 20 + 100 = 120, \ldots$ $100^2 = 10000 > 100 + 100 = 200$. According to a common sense judgment, x^2 is much larger than $x + 100$.

Even more striking difference is provided by yet another canonical example: $n^2 + 100$ versus 2^n. Here is the table of the initial values of those functions.

n	$n^2 + 100$	2^n
1	101	2
2	104	4
3	109	8
$\ldots$	$\ldots$	$\ldots$
10	200	1024
$\ldots$	$\ldots$	$\ldots$
100	10100	$> 10^{30}$

Note that for $n = 100$ (a moderate size input), $n^2 + 100$ constitute a negligible fraction of 2^n, something like 10^{-25}. The main reason for such a mismatch is, of course, the exponential function 2^n which becomes ridiculously large. In computational complexity, those algorithms which make exponential (of the size of input) number of steps are regarded **non-feasible**.

In what follows f, g, h may be regarded as functions from $\mathbf{R}$ to $\mathbf{R}$ (as well as from to $\mathbf{N}$ to $\mathbf{R}$). Reminder: $|a|$ stands for the absolute value of a when the sign of a is stripped off. Formally speaking, $|a| = a$, if $a \geq 0$ and $|a| = -a$, if $a < 0$. For example, $|2| = 2$, $|-2| = -(-2) = 2$, $|0| = 0$. Some common inequalities which follow immediately from the definitions: $a \leq |a|$, $|a \pm b| \leq |a| + |b|$.

Definition 6.1. $f = O(g)$ (read *f is big-O of g*) if $\exists C \exists k \forall x(x > k \to (|f(x)| \leq C \cdot |g(x)|)$. The informal meaning of "f is $O(g)$" is that some multiple of g eventually overruns f. This relation is usually regarded a sort of inequality on functions $f \preceq g$. We say that f and g have the **same order** (notation $f = \Theta(g)$), of both $f = O(g)$ and $g = O(f)$.
Examples:

- $x + 100$ is $O(x^2)$. Indeed, put $C = 1, k = 100$ and $x > k$. Then $x + 100 < x \cdot 100 < x^2$.
- $x^2 + 100x$ is $O(x^2)$ (despite the fact that the former functions is always greater than the latter for positive xs, moreover, their difference $(x^2 + 100x) - x^2 = 100x$ grows to $+\infty$). Indeed, put $C = 2, k = 100$. Then $x > 100 \;\to\; x^2 + 100x < x^2 + x \cdot x = 2x^2$.
- x^2 is $O(x^2 + 100x)$ $(C = 1, k = 0)$, therefore, f and g have the same order.
- $x + 100$ and x^2 do not have the same order, on particular, x^2 is not $O(x + 100)$. Indeed, let us first negate the definition of "big-O" above: $\forall C \forall k \exists x(x > k \wedge (|f(x)| > C \cdot |g(x)|)$, and then prove that the pair $f(x) = x + 100$ and $g(x) = x^2$ satisfy the negation of the "big-O" definition. Let C, k are any given reals. Take any $x > \max(k, 2C, 100)$. Then

$$x^2 > 2Cx = C \cdot 2x = C(x + x) > C(x + 100).$$

Example 6.2. Some simple reference functions of n used in the complexity theory. Here $f \prec g$ means "*$f = O(g)$ but not $g = O(f)$*", $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, log is the base 2 logarithm.

$$1 \prec \log\log n \prec \log n \prec n \prec n \log\log n \prec n \log n \prec n^2 \prec n^3 \prec \ldots \prec 2^n \prec 3^n \prec \ldots \prec n! \prec n^n$$

Theorem 6.3. *If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 + f_2 = O(\max(|g_1|, |g_2|))$.*
Proof. For appropriate C_1, k_1, C_2, k_2 we have

$x > k_1 \Rightarrow |f_1(x)| \leq C_1|g_1(x)| \quad x > k_2 \Rightarrow |f_2(x)| \leq C_2|g_2(x)|$.

Without loss of generality we assume that $C_1, C_2 > 0$. Put $k = \max(k_1, k_2)$, $C = C_1 + C_2$, $g = \max(|g_1|, |g_2|)$. Then $x > k \Rightarrow |f_1(x) + f_2(x)| \leq |f_1(x)| + |f_2(x)| \leq C_1|g_1(x)| + C_2|g_2(x)| \leq C_1|g(x)| + C_2|g(x)| \leq (C_1 + C_2)|g(x)| = C|g(x)|$.
Example: $x^2 = O(x^2)$, $100x = O(x^2)$, therefore $x^2 + 100x = O(x^2)$.

Theorem 6.4. *If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 \cdot f_2 = O(g_1 \cdot g_2)$.*
Proof. For appropriate C_1, k_1, C_2, k_2 we have

$x > k_1 \Rightarrow |f_1(x)| \leq C_1|g_1(x)| \qquad x > k_2 \Rightarrow |f_2(x)| \leq C_2|g_2(x)|$.

Put $k = \max(k_1, k_2)$, $C = C_1C_2$. Then $x > k \Rightarrow |f_1(x)f_2(x)| = |f_1(x)| \cdot |f_2(x)| \leq C_1|g_1(x)| \cdot C_2|g_2(x)| = C_1C_2|g_1(x) \cdot g_2(x)| = C|g_1(x) \cdot g_2(x)|$.
Example: Give as good big-O estimate as possible in terms of simple reference functions for $f(n) = (3n+1)\log(5n^3 + 1) + 10n^2$. We may assume that n is sufficiently large. $3n+1 = O(n)$, $\log(5n^3 + 1) < \log(6n^3) = \log 6 + 3\log n < 4\log n = O(\log n)$ (this works for $n > 6$). Therefore, $(3n + 1)\log(5n^3 + 1) = O(n \log n)$. Since $10n^2 = O(n^2)$, and $\max(n \log n, n^2) = n^2$, by 6.3, $f(n) = O(n^2)$. Moreover, since $n^2 < f(n)$ we have $n^2 = O(f)$ and thus $f(n) = \Theta(n^2)$.

Theorem 6.5. *$F(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ $(a_n \neq 0)$ has order x^n.*
Proof. We show $f(x) = O(x^n)$ and leave $x^n = O(f)$ as an exercise. Let $k = 1$ and $x > k$. Then $|a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0| \leq |a_n|x^n + |a_{n-1}|x^{n-1} + \ldots + |a_1|x + |a_0| \leq x^n(|a_n| + \frac{|a_{n-1}|}{x} + \ldots + \frac{|a_1|}{x^{n-1}} + \frac{|a_0|}{x^n}) \leq x^n(|a_n| + |a_{n-1}| + \ldots + |a_1| + |a_0|)$. Now put $C = |a_n| + |a_{n-1}| + \ldots + |a_1| + |a_0|$ and get the desired $|f(x)| \leq C \cdot x^n$ whenever $x > k$.

Homework assignments. (due Friday 02/09).

A. Section 1.8: 2, 8ab, 20ab, 28a

B. Show that $2^n = O(n!)$, but $n!$ is not $O(2^n)$.