1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 1.7
2. The main message of this lecture:

> **Summation of sequences, both finite and infinite, was one of the first recursive procedures ever. In some classical cases: e.g. arithmetic, geometric progressions, it was possible to find concise algebraic formulas.**

A naive understanding of a sequence is some sort of an infinite ordered list of elements $a_0, a_1, a_2, \ldots$. Using the notion of function we can give a more scientific definition.

Definition 5.1. A **sequence** is a function f whose domain is the set $\mathbf{N}$ of natural numbers $\{0, 1, 2, \ldots\}$. For the sake of tradition we denote $f(n)$ by a_n and write a sequence in the form $\{a_n\} = a_0, a_1, a_2, \ldots$. We will also be considering sequences starting from 1 (functions having domain $\{1, 2, 3 \ldots\}$), or even from some other integer. Examples:

$$\begin{array}{ll} \{n\} = 0, 1, 2, \ldots & \{n^2\} = 0, 1, 4, 9, 16, 25, \ldots \\ \{1/n\}_{(n \geq 1)} = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots & \{2^n\} = 1, 2, 4, 8, 16, \ldots \\ \{1/2^n\} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots & \{(-1)^n\} = 1, -1, 1, -1, 1, -1, \ldots \end{array}$$

Mind the difference between a sequence $a_0, a_1, a_2, a_3, \ldots$ and a set $\{a_0, a_1, a_2, a_3, \ldots\}$. For example, the sequence $\{(-1)^n\} = 1, -1, 1, -1, 1, -1, \ldots$ is infinite, whereas the set
$\{1, -1, 1, -1, 1, -1, \ldots\} = \{1, -1\} = \{-1, 1\}$.

Definition 5.2. A **string** is a finite sequence $a_0, a_1, a_2 \ldots, a_n$ (or $a_1, a_2, \ldots, a_n$), which can be identified with a function having domain $\{0, 1, 2, \ldots, n\}$ (or $\{1, 2, \ldots, n\}$ respectively).

Definition 5.3. Summation notation:

$$a_m + a_{m+1} + \ldots + a_n = \sum_{i=m}^{n} a_i$$

Here a_i is the **summation term**, i is the **index of summation**, m is the **lower limit** and n is the **upper limit** of the summation. Examples:

$$1 + 2 + 3 + \ldots + 10 = \sum_{i=1}^{10} i = 55 \qquad 1 + 4 + 9 + \ldots + 100 = \sum_{i=1}^{10} i^2 = 385$$

Note that the summation index i is a *bound variable* and thus can be renamed with changing the meaning of the summation. More examples:

$$\sum_{i=1}^{10} a_i = \sum_{k=1}^{10} a_k, \quad \sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i, \quad \sum_{i=m}^{n} \lambda a_i = \lambda \sum_{i=m}^{n} a_i, \quad \sum_{i=1}^{n} c = nc$$

Definition 5.4. An **arithmetic progression** is a sequence satisfying $a_{n+1} - a_n = d$, where d is constant for a given progression, called the **difference** of the progression. An arithmetic progression is $a, a + d, a + 2d, a + 3d, \ldots = \{a + nd\}$ and thus is specified by d and its initial term a. Examples:

$1, 2, 3, 4, \ldots$ (here $a = 1$, $d = 1$)
$100, 50, 0, -50, -100, \ldots$ ($a = 100$, $d = -50$)
$-2, 0, 2, 4, 6, \ldots$ (here $a = -2$, $d = 2$)
$\{n^2\} = 0, 1, 4, 9, 16, 25, \ldots$ ($a = 0$, $d =$? sorry, it is *not* an arithmetic progression!).

Theorem 5.5. *The sum $S_n = a_1 + a_2 + \ldots + a_n$ of the first n terms of an arithmetic progression is $S_n = n(a_1 + a_n)/2$.*

Proof. Count S_n twice as $2S_n = (a_1 + a_2 + \ldots + a_{n-1} + a_n) + (a_n + a_{n-1} + \ldots + a_2 + a_1)$, take the sums of the corresponding terms in both of those expressions: $2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1)$. Note that the sum of terms inside each of n pairs of parentheses is constant, and thus equals to the first of them $(a_1 + a_n)$. Therefore, $2S_n = n \cdot (a_1 + a_n)$ and $S_n = n \cdot (a_1 + a_n)/2$.

Examples: $1 + 2 + 3 + \ldots + 10 = (1 + 10) \cdot 10/2 = 11 \cdot 10/2 = 11 \cdot 5 = 55$,

$1 + 3 + 5 + \cdot + 15 = (1 + 15) \cdot 8/2 = 16 \cdot 4 = 64$ (which is a whole square. Is it a coincidence?)

Definition 5.6. A **geometric progression** is a sequence satisfying $a_{n+1}/a_n = r$, where $r \neq 0$ is constant for a given progression called the **common ratio**. The geometric progression with an initial term $a \neq 0$ and a common ratio r is $a, ar, ar^2, ar^3, \ldots$, i.e. $a_n = a \cdot r^{n-1}$.

Examples: $\{2^n\}$ (here $a = 1$, $r = 2$), $\{2^{-n}\}$ (here $a = 1$, $r = \frac{1}{2}$),

$1, -1, 1, -1, 1, -1, \ldots = \{(-1)^n\}$ (here $a = 1$, $r = -1$),

$\{n^2\}$ (sorry, it is *not* a geometric progression, since $3^2/2^2 = 2\frac{1}{4} \neq 4 = 2^2/1^2$).

Theorem 5.7. *The sum $S_n = a_1 + a_2 + \ldots + a_n$ of the first n terms of a geometric progression with common ratio $r \neq 1$ is $S_n = (a_{n+1} - a_1)/(r - 1)$. If $r = 1$ then $S_n = na$.*

Proof. Let us first evaluate the sum a simple geometric progression $1 + r + r^2 + \ldots + r^{n-1}$ where $r \neq 1$. Multiply this sum by $(1 - r)$:

$$(1 + r + r^2 + \ldots + r^{n-1}) \cdot (1 - r) = 1 + r + r^2 + \ldots + r^{n-1} - r - r^2 - r^3 - \ldots - r^{n-1} - r^n.$$

In this long sum all the terms but the first and the last ones cancel (!), therefore it is equal to $1 - r^n$, therefore

$$(1 + r + r^2 + \ldots + r^{n-1}) \cdot (1 - r) = 1 - r^n.$$

Divide both parts by $(1 - r)$, which is not 0, and conclude that

$$1 + r + r^2 + \ldots + r^{n-1} = (1 - r^n)/(1 - r) = (r^n - 1)/(r - 1).$$

Let us now consider the general case: $S_n = a_1 + a_2 + \ldots + a_n = a + ar + ar^2 + \ldots + ar^{n-1} = a(1 + r + r^2 + \ldots + r^{n-1}) = a(r^n - 1)/(r - 1) = (a_{n+1} - a_1)/(r - 1)$.

Examples: $1 + 2 + 4 + 8 + \ldots 2^n = (2^{n+1} - 1)/(2 - 1) = 2^{n+1} - 1$,

$\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = (\frac{1}{2^{n+1}} - \frac{1}{2})/(\frac{1}{2} - 1) = (\frac{1}{2} - \frac{1}{2^{n+1}})/(1 - \frac{1}{2}) = 2(\frac{1}{2} - \frac{1}{2^{n+1}}) = 1 - \frac{1}{2^n}$

Definition 5.8. Double summation is a sum of a two-dimensional array $\{a_{ij}\}$:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = (a_{11} + a_{12} + \ldots + a_{1n}) + (a_{21} + a_{22} + \ldots + a_{2n}) + \ldots + (a_{m1} + a_{m2} + \ldots + a_{mn})$$

Example:

$$\sum_{i=1}^{3} \sum_{j=1}^{4} (i^2 - j) = [(1 - 1) + (1 - 2) + (1 - 3) + (1 - 4)] +$$

$$[(4 - 1) + (4 - 2) + (4 - 3) + (4 - 4)] + [(9 - 1) + (9 - 2) + (9 - 3) + (9 - 4)] = 26$$

Homework assignments. (The first installment due Friday 02/09).

A. Section 1.7: 2, 10acf, 16b, 18cd

B. Find $S = 1 + 3 + 5 + \ldots + 99$ using the summation formula for arithmetic progressions. Is S a whole square?