1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 1.6
2. The main message of this lecture:

> **The word "function" came from its Latin equivalent *functio*, which means "execution". It was introduced in mathematics in the late 17th century by Leibniz. Euler used it to describe equations involving variables and constants. The modern definition is much broader, and covers arbitrary sets, including the ones important for Computer Science: integers, strings, graphs, etc.**

Definition 4.1. Let A and B be arbitrary sets. A **function** f is a rule that assigns each element $a \in A$ exactly one element $b \in B$. The element b is called the **image** (or **value**) of a under f and is denoted by $f(a)$ (read: "f of a"). The set A is called the **domain** of f. The set B is called **codomain** of f. For any $S \subseteq A$ the set $f(S) = \{y \mid y = f(x) \textit{ for some } x \in S\}$ is called the **image** of S. $f(A)$ is called **range** of f. To define a function f, we use the notation

$$f:\ A \to B$$

Comment 4.2. The words "a rule that assigns..." in the above definition should be read in the most general sense, similar to "property" or "predicate". A priori, there are no requirements for a function to be specified by some sort of a formula or to be computable. For the most abstract definition of a function see 4.6 below. Functions defined above are *total* (f defined on the whole A) and *single valued* (there is at most one value b for each $a \in A$). In many areas of Computer Science there is a need to consider also non total (partial) or many valued functions. It is also unfortunate that the mapping symbol $\to$ coincides with the implication symbol $\to$ from logic. However, such a collision of notations is not really a danger. Usually it does not take Einstein to distinguish which of those symbols we are dealing with in any particular situation.

Other common notations concerning functions: $x \to f(x)$ or $y = f(x)$. Those notations are usually applied when there is a nice formula describing $f(x)$, for example $x \to x^2$ or $y = x^2$ for the squaring function.

Consider all functions of type $f : A \to B$. Any operation on B can be naturally extended to the corresponding operation on f's. The standard example is provided by $B = \mathbf{R}$, i.e. the set of reals, with addition"+" and multiplication "$\cdot$". For any two functions $f, g : A \to \mathbf{R}$ we can define $f + g$ (the sum of f and g) and $f \cdot g$ (the product of f and g). To define the function $h = f + g$ we have to specify exactly where arbitrary $x \in A$ is mapped to by h (similar for $f \cdot g$). The standard way of doing that is by the identities

$$(f + g)(x) = f(x) + g(x),$$
$$(f \cdot g)(x) = f(x) \cdot g(x).$$

A standard example: $f(x) = x + 1$, $g(x) = x + 2$. Then

$$(f + g)(x) = (x + 1) + (x + 2) = 2x + 3 \qquad (f \cdot g)(x) = (x + 1)(x + 2) = x^2 + 3x + 2.$$

A less standard example: a piece x of luggage admitted for transatlantic flights without extra pay should have the sum of three dimensions length $l(x)$, width $w(x)$ and hight $h(x)$ not greater than 150 in. This condition may be presented by a formula $(l + w + h)(x) \leq 150$. Note that there is no need to have any operations on A here.

Definition 4.3. Let now $g : A \to B$ and $f : B \to C$. The **composition** $f \circ g$ of the functions f and g is defined as the result of consecutive execution of g and f: $(f \circ g)(x) = f(g(x))$. This notation is a bit counterintuitive. The right-hand side $f(g(x))$ of this equation specifies that given $x \in A$ one has first to find $y = g(x)$ and then $f(y) = f(g(x))$, whereas the left-hand side of this equation sort of suggests performing f first... Do not follow the latter!

Example: Let $f(x) = x + 1$ and $g(x) = x + 2$. Then $(f \circ g)(x) = (x + 2) + 1 = x + 3$.

Note that the composition $f \circ g$ is defined for any pair of functions f and g for which the range of g is a subset of the domain of f. No special operations on those sets are required. A very important property of the functions composition "$\circ$": it is not commutative, i.e. generally speaking $f \circ g \neq g \circ f$. Example: $f(x) = x + 1$, $g(x) = x^2$. Then $(f \circ g)(x) = x^2 + 1$ whereas $(g \circ f)(x) = (x + 1)^2 = x^2 + 2x + 1$. Do not confuse $f \circ g$ with $f \cdot g$. By the way, the latter is commutative $f \cdot g = g \cdot f$ as soon as the usual multiplication is.

Definition 4.4. A function $f : A \to B$ is **one-to-one** (or **injective**), if $f(x_1) = f(x_2)$ yields $x_1 = x_2$. f is **onto** (or **surjective**) if $B = f(A)$. If f is both one-to-one and onto, it is called **one-to-one correspondence** (or **bijection**). Examples (on reals): $y = e^x$ is one-to-one, but not onto; $y = x^3 - x$ is onto, but not one-to-one; $y = x^3$ is both. i.e. is one-to-one correspondence. For more examples see the textbook and the slides.

Unofficially, the inverse function to f reverses an effect of f, i.e. if $y = f(x)$, then $f^{-1}(y) = x$. Since here we admit single valued and total functions only, not every f has an inverse. For example, consider the squaring function from $\mathbf{R}$ to $\mathbf{R}$ $f(x) = x^2$, e.g. $f(2) = 4$. A hypothetical inverse function here f^{-1} should then map 4 back to 2: $f^{-1}(4) = 2$. However, f is not one-to-one, in particular, $f(-2) = 4 = f(2)$. Therefore, $f^{-1}(4)$ should be equal to -2 which means f^{-1} does not really exist. Another difficulty with the inverse here is caused by our careless assumption that f operates from $\mathbf{R}$ to $\mathbf{R}$ which makes f not onto. After that we have no choice but expecting f^{-1} to be defined on the whole of $\mathbf{R}$ too. But what would be $f^{-1}(-4)$ then? It is easy to see that if we fix both of the obstacles: not one-to-one and not onto (for example, by limiting f to **nonnegative reals** $\mathbf{R}^+$ only), a function f becomes invertible.

Theorem 4.5. *Let $f : A \to B$ be a one-to-one correspondence. Then there exists a unique function $f^{-1} : B \to A$ called the* **inverse** *of f such that $f^{-1}(b) = a$ whenever $f(a) = b$.*

Proof. For each $b \in B$ define $f^{-1}(b)$ as such $a \in A$ that $f(a) = b$. Then f^{-1} is defined on the whole of B (since the original function f was onto B), and single valued, since f was one-to-one.

An important observation: $f^{-1} \circ f = \iota_A$, i.e. the identity function on A. Likewise, $f \circ f^{-1} = \iota_B$. Furthermore, $(f^{-1})^{-1} = f$. i.e. the inverse of the inverse is the original function.

Definition 4.6. Let $f : A \to B$. The **graph** of f is the set of pairs $\{(a, f(a)) \mid a \in A\}$ (see many examples on the slides and in the textbook). In fact, the official mathematically rigid way to introduce the notion of function in set theory is via graphs: function is a set of pairs $F = \{(a, b) \mid a \in A \wedge b \in B\}$ which is total and single valued $\forall a \exists! b((a, b) \in F)$.

Homework assignments. (The last installment due Friday 02/02).

A. Section 1.6: 16, 44, 48

B. Find $f \circ g$ and f^{-1} when $f(x) = x^3 - 1,\ \ g(x) = x + 1$ on $\mathbf{R}$.