CS280, Spring 2001	**Handout 33**
April 18, 2001	

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 7.7, 7.8
2. The main message of this lecture:

> **Planar graphs and coloring provide two extreme examples of problems in mathematics. The former admits a complete and elegant human made theory spanning from the 18th to the early 20th centuries. The latter produced the first example of a computer solution of a top rated problem in theoretical mathematics: the four color problem. Note that despite continuous efforts no human proof of this problem has been discovered yet.**

Definition 33.1. A graph is called **planar** if it can be drawn in the plane without any edge crossing. A more strict mathematical definition not based on a vague notion of "drawing" is too tedious to work with. Besides, our intuition of planarity is so clear that in a practical sense we do not need a more formal definition. Examples: K_4 is planar though its usual picture as a square with diagonals does not look planar: it can be made one by drawing one of the diagonals outside the square (cf. the slides). Another good example of a planar graph is Q_3: its standard visualization is a 3D cube, whereas its clearly admits a planar representation as well (slides!).

Example 33.2. The standard example of a non-planar graph is called "three houses three utilities". Speaking mathematically, it is a complete bipartite graph $K_{3,3}$, where the first partition set represents three houses, the second one three utilities (gas, water, electricity) and edges are connecting utility lines which we do not want to intersect. Unfortunately for the utility engineers $K_{3,3}$ is not planar (slides!), therefore there is no a planar connection scheme without edges crossing.

Theorem 33.3. (Euler's formula) *In a simple connected planar graph G let e be the number of edges, v the number v of vertices, and r the number of regions the graph split the plane into. Then $r = e - v + 2$.*

Proof. Induction on the number e of edges in G.
BASE. $e = 1$, i.e. $G = K_2$. Then $v = 2$, $r = 1$, which yields $e - v + 2 = 1 - 2 + 2 = 1 = r$.
INDUCTION HYPOTHESIS. The formula $r = e - v + 2$ holds for any graph of the above kind having $e = n$ edges.
INDUCTION STEP. Consider a graph G of the above kind having $e = n + 1$ edges. Remove one edge (and possibly a vertex) without violating the connectivity property by the following procedure. Pick a path P in G of maximum possible length without repeating edges. Since G is finite, the path P either has a loop, or terminates in a suspended vertex. In the former case remove an edge in a loop, in the latter remove the last edge and the last vertex (see slides). The remaining graph G' already has n edges and thus falls under the I.H.

Case A. Only an edge is removed. Then in $e' = e - 1$, $v' = v$, $r' = r - 1$, since the removed edge from a loop separated two distinct areas in a planar graph G (see slides). By the I.H., $r' = e' - v' + 2$, thus $r - 1 = (e - 1) - v + 2$, and $r = e - v + 2$.

Case B. An edge and a vertex are removed. Then the removed edge was a dead end in G not separating any regions. In this case $e' = e - 1$, $v' = v - 1$, $r' = r$. By the I.H., $r' = e' - v' + 2$, thus $r = (e - 1) - (v - 1) + 2$, and $r = e - v + 2$.

Example 33.4. In K_3: $r = 2$, $e = v = 3$, $e - v + 2 = 3 - 2 + 2 = 2 = r$. In K_4: $r = 4$, $e = 6$, $v = 4$, $e - v + 2 = 6 - 4 + 2 = 4 = r$. In K_5: $v = 5$, $e = 10$, $r = \ldots$ Oops, this graph is not planar!. Below you can find a proof of this fact.

Corollary 33.5. (A useful test of non-planarity. Its power is based on the fact that it relates the number of edges and the number of vertices without involving the number of regions which might not exist) *In a connected simple planar graph with the number of vertices $v \geq 3$ the following inequality holds: $e \leq 3v - 6$.*
Proof. Walk along the boundary of any region R one counting the number of edges passed. The resulting number is called the degree of R. Note that dead-ends inside R and similar edged contribute 2 to the degree! (see slides). Clearly, the degree of each region (including the unbounded one) is at least 3. Therefore, The sum S of degrees over all regions is at least $3r$. On the other hand $S = 2e$, since every edge is traversed and counted in S exactly twice. Therefore $2e = S \geq 3r$. By Euler's formula, $2e \geq 3(e - v + 2)$. Hence $2e \geq 3e - 3v + 6$, and $e \leq 3v - 6$.

Examples 33.6. K_5 is not planar: $v = 5$, $e = 10$, $3v - 6 = 15 - 6 = 9 < 10 = e$. Proving that $K_{3,3}$ is not planar required another effort of the same kind (see 7.7 in the book).

Theorem 33.7. (Kuratowski's theorem providing the ultimate criterion of planarity) *A graph is non-planar if and only if it contains either $K_{3,3}$ or K_5.*
Comments. (instead of a proof). Here "contains" means "contains a subgraph homeomorphic to". In turn, "G' is homeomorphic to G''" means "G' and G'' are obtained from the same G by adding new vertices on already existing edges (see slides).

Bonus problem 33.8. Is Q_4 planar? Submit solutions in written to the instructor.

Definition 33.9. A **coloring** of a simple graph is the assignment of a color to each vertex so that no two adjacent ones get the same color. The **chromatic number** of a graph is the least number of colors needed for its coloring.

Examples 33.10. Scheduling problem reduces to coloring. How many time slots are needed to schedule the finals so that no student has two exams in the same time? Consider a graph whose vertices represent courses, and two vertices are connected by an edge if they have some students in common. Then the time slots can be represented by colors, and the number of time slots needed is exactly the chromatic number of the resulting graph.

Examples 33.11. The chromatic number of K_n is n, since every two vertices are adjacent and thus have to be colored differently. The chromatic number of $K_{m,n}$ is 2 (and thus does not depend on m, n. Indeed, paint all the elements of the partition set one red, and all elements of the partition set two blue.

Theorem 33.12. *A chromatic number of a planar graph is no greater than four.*
This was one of the most famous problems in mathematics, standing since the 1850s with quite a history of fallacious proofs. It was finally proved in 1979 by Appel and Haken using about 1000 hours of a powerful computer time to perform an exhausting search at the final stage of the proof. No human proof of this problem has been found yet.

Homework assignments. (The first installment due Friday 04/27)

33A:Rosen7.7-4; 33B:Rosen7.7-6; 33C:Rosen7.7-12 33D:Rosen7.7-18 33E:Rosen7.8-4 33F:Rosen7.8-16.