CS280, Spring 2001	**Handout 31**
April 13, 2001	

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 7.4, 7.5

2. The main message of this lecture:

   Walking along edges in graphs is fun. It is also a major source of both theoretical and practical problems in this area.

Definition 31.1. A **path of length** n **from** u **to** v in a simple graph G is a sequence of vertices $u = x_0, x_1, x_2, \ldots, x_n = v$ such that $\{x_i, x_{i+1}\}$ in an edge.

In undirected graphs such a path is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $f(e_i) = \{x_{i-1}, x_i\}$, $f(e_{i+1}) = \{x_i, x_{i+1}\}$, $x_0 = u$, $x_n = v$.

In directed graphs such a path is is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $f(e_i) = (x_{i-1}, x_i)$, $f(e_{i+1}) = (x_i, x_{i+1})$, $x_0 = u$, $x_n = v$.

In all those cases a path is nothing but a sequence of back-to-back edges which takes into account directions of edges, if any. Paths are easy to visualize, provides a graph has a geometric representation as a set of nodes and arcs (arrows).

Definition 31.2. A path from u to v is a **circuit (cycle)** if $u = v$. A **simple path** does not contain the same edge more then once.

Definition 31.3. An graph is **connected** if there is a path between every pair of distinct vertices. For directed graphs there is an additional notions of *weak connectivity*: a directed graph is **weakly connected** if any pair of distinct vertices is connected by a path *not necessarily obeying the directions of edges*. For examples see slides/textbook.

Some new isomorphic invariants that help to rule out isomorphism of graphs: connectivity, existence of a simple circuit of a particular length, etc.

Definition 31.4. A **Euler path** in a graph G is a simple path containing every edge of G. A **Euler circuit** is a Euler path which is a circuit. i.e. begins and ends at the same vertex. (cf. the Seven Bridges of Königsberg puzzle).

Theorem 31.5. *A finite connected multigraph has a Euler circuit if and only if each of its vertices has even degree.*

Proof. Suppose G has a Euler circuit. Let us walk along its edges. Each time the path comes into a vertex it has to leave the vertex *by the different edge*, since we are not allowed to step on the same edge twice. Since the path is closed (a circuit!) there are no initial nor terminal vertices and each time we enter a vertex we have to leave it. Therefore, the number of incoming edges is equal to the number of outcoming edges for each vertex, the sum of the number of edges for each vertex is then even.

Let now G have even number of edges by each vertex. Here is a neat construction that builds a Euler circuit for G. Pick an arbitrary vertex u, run a Euler path from this vertex as long as possible (i.e. without repeating edges). Sooner or later we will stuck because the total number of edges is finite and we cannot run long without repetitions. The important observation is that we can stop only in the initial vertex u! Indeed, u is the only vertex in G which we left without entering first; any other vertex when entered has at least one free edge out. If we have already visited all the edges, we are done. Otherwise delete all the edges visited and all the vertices that became isolated. The remaining subgraph H still has vertices of even degrees only, since we deleted edges by pairs of incoming-outcoming ones. Note that H is not necessarily connected. However, each remaining connectivity unit has at least one vertex v in common with the deleted circuit. Pick v as the next starting point for building a Euler circuit in H. The next important observation is that we can splice a new circuit on H with the removed circuit: start at v, walk the old circuit and return to v, then proceed from v in the new circuit returning back to v. Now remove the new circuit as before, and proceed along those lines until the whole G is removed, i.e. covered by one compound Euler circuit.

Theorem 31.6. *A finite connected multigraph has a Euler path but not a Euler circuit if and only if it has exactly two vertices of odd degree.*
Proof. Basically the same construction as in 31.5. One of the odd degree vertices is the starting point of a Euler path, the other is the end point.

Definition 31.7. A **Hamilton path** in a graph G is path that visits every vertex exactly once. A **Hamilton circuit** is a Hamilton path which is a circuit. See many examples in slides/textbook.

There is no easy way of finding whether a graph G has a Hamilton path. It is an *NP*-complete problem.

Homework assignments. (The second installment due Friday 04/20)

31A:Rosen7.5-6; 31B:Rosen7.5-24; 31C:Rosen7.5-42.