1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 7.3
2. The main message of this lecture:

Matrices are a universal mechanism of representing graphs.

Definition 30.1. Let G be a finite graph or any kind (simple graph, multigraph, pseudograph, both directed or undirected) with the set of vertices $V = \{v_1, v_2 \dots, v_n\}$. The **adjacency matrix** A_G of $G = (V, E)$ is an $n \times n$ matrix $[a_{ij}]$ whose entries a_{ij} are the number of edges going from v_i to v_j. The convention is that an undirected edge $\{v_i, v_j\}$ counts both as going from v_i to v_j and as going from v_j to v_i.

It is immediate that the entries of an adjacency matrix are nonnegative integers. In the case of *simple graph* G the adjacency matrix A_G is a bit matrix with the entries

$$a_{ij} = \begin{cases} 1, & \text{if there } \{v_i, v_j\} \text{ is an edge} \\ 0, & \text{otherwise} \end{cases}$$

Moreover, the diagonal entries $a_{ii} = 0$ for all $i = 1, 2, \dots, n$ (reflecting the fact that G has no loops), and A_G is symmetric, i.e. $a_{ij} = a_{ji}$ (since $\{v_i, v_j\} = \{v_j = v_i\}$.

Example 30.2. The adjacency matrix of the cycle C_3 is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

For a *multigraph* G (parallel edges are allowed) the entries of the adjacency matrix may be any nonnegative integer, but the diagonals are still 0's and A_G is still symmetric. For a *pseudograph* G the diagonal entries are not necessarily zero, but A_G is still symmetric. For a directed graph the symmetry of A_G is no longer required.

Example 30.3. Draw a (directed ...)graph G having the adjacency matrix A_G below:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(the answer is on lecture slides)

Definition 30.4. **Adjacency list** of a graph G without multiple edges is the list specifying the vertices that are adjacent to each vertex of G. For example, the following adjacency matrix and adjacency list represent the same directed graph

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Initial Vertex	Terminal Vertices
a	a, b, c
b	a, c
c	b, c

Adjacency lists are practical for **sparse** adjacency matrices, that is, matrices with few nonzero entries.

Definition 30.5. Let G be a finite undirected graph (multigraph, pseudograph) with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. The **incidence matrix** of G is a bit $n \times m$ matrix $[m_{ij}]$ with the entries

$$m_{ij} = \begin{cases} 1, & \text{when edge } e_j \text{ is incident with } v_i \\ 0, & \text{otherwise} \end{cases}$$

Example 30.6. The following incidence matrix represents the wheel W_4:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The next notion we have to discuss is the **graph isomorphism**. Before considering a formal definition below let us remember that there is a good intuition of the graph isomorphism: two graphs are isomorphic if renaming vertices (if necessary) of one of them can make it equal to the other. Example: the complete graph K_4 with the vertices a_1, a_2, a_3, a_4 is isomorphic to the wheel W_3 with the peripheral elements p_1, p_2, p_3 and the center c. Here any renaming works, e.g. $a_1 \mapsto p_1$, $a_2 \mapsto p_2$, $a_3 \mapsto p_3$, $a_4 \mapsto c$. Another classical example: the cycle C_5 and the 5-star.

Definition 30.7. Two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a one-to-one correspondence f between vertices of the two graphs that preserves the adjacency relationship: a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2.

Establishing isomorphism of graphs is a computational hard problem since there are $n!$ distinct one-to-one correspondences between two sets of vertices of cardinality n. However, there are some simple tricks that help establishing *non-isomorphism* of given graphs. The generic name for those tricks is **invariants**, i.e. the properties of graph that should be preserves under isomorphism, if any. Examples of those invariants for simple graphs are: the number of vertices, the number of edges, the set of possible degrees of vertices, the number of vertices having a given degree, etc.

Homework assignments. (The first installment due Friday 04/20)

30A:Rosen7.3-4; 30B:Rosen7.3-12; 30C:Rosen7.3-20; 30D:Rosen7.3-38; 30E:Rosen7.3-42.