1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 1.4, 1.5
2. The main message of this lecture:

> **Whereas the language of logic provides a framework for *formalized mathematics*, the traditional mathematics uses the language of sets. There is a close connection between predicates and sets where logical connectives define simple operations on sets.**

There is no an exact definition of a *set*. Moreover, taken it its full generality, Cantor's "naive" set theory adopted in this course would lead to a contradiction. However, we will be dealing with specific well-defined sets which are free of set theoretical paradoxes, and we will not try to push the limits of abstraction. Mathematicians have figured out how to do this right via axioms systems, but you would not like the answer...

Set is an arbitrary collection of objects, called elements, which can well itself be sets, etc. Notation $x \in A$ (*x is an element of A*). More notations:

$\{a_1, a_2, \ldots, a_n\}$	the set with elements $a_1, a_2, \ldots, a_n$
$\{x \mid A(x)\}$	the set of elements satisfying property $A(x)$
$\emptyset$	"empty set". Here $A(x)$ is the constant *false* $\mathbf{F}$.

In particular, $A = \{x \mid x \in A\}$ for each set A.

Sets are completely determined by their elements. By so-called *Correspondence Principle*, equivalent logical conditions specify the same set of objects

$$\textit{If}\ \ A(x) \Leftrightarrow B(x)\ \ \textit{then}\ \ \{x \mid A(x)\} \ = \ \{x \mid B(x)\}.$$

In particular, $\emptyset$ is the set of objects satisfying a contradictory condition.

Example: $\{x \mid x \textit{ in a prime integer less than } 10\} = \{2, 3, 5, 7\}$.

Definition 3.1. U is a **subset** of V (notation $U \subseteq V$) if every element of U is an element of V. U is a **proper subset** of V (notation $U \subset V$) if $U \subseteq V$, but $U \neq V$.

Examples: $\{1, 3, 5\} \subset \{1, 2, 3, 4, 5\}$, *integers* $\subset$ *rationals* $\subset$ *reals.*

If U and V are specified by conditions, i.e. $U = \{x \mid A(x)\}$ and $V = \{x \mid B(x)\}$, then the inclusion $U \subseteq V$ means that $\forall x(A(x) \to B(x))$ holds. In particular, $\emptyset \subseteq U$ for any U. Indeed, since $\mathbf{F} \to P(x)$ for every predicate $P(x)$, by the Correspondence Principle,
$\{x \mid \mathbf{F}\} \subseteq \{x \mid x \in U\}$, i.e. $\emptyset \subseteq U$.

Definition 3.2. The **power set** of U (notation $P(U)$) is the collection of all subsets of U, including $\emptyset$ and U itself. Example: $P(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, $P(\emptyset) = \{\emptyset\}$.

Mind the difference between $\in$ and $\subseteq$: $1 \in \{0, 1\}$, but $1 \notin P(\{0, 1\})$. $\{1\} \notin \{0, 1\}$, but $\{1\} \subseteq \{0, 1\}$ and $\{1\} \in P(\{0, 1\})$. Note that $\emptyset \notin \{0, 1\}$, but $\emptyset \subseteq \{0, 1\}$, $\emptyset \in P(\{0, 1\})$ and $\emptyset \subset P(\{0, 1\})$.

Definition 3.3. If a set X has exactly n distinct elements then we say that the **cardinality** of X is n (notation $|X| = n$). In such a case X is called a **finite** set.

Theorem 3.4. *If* $|X| = n$ *then* $|P(X)| = 2^n$.

Proof. Let $X = \{a_1, a_2, \ldots, a_n\}$. Each $Y \subseteq X$ is uniquely specified by its characteristic string of bits C_Y of length n having i-th entry 1 if $a_i \in Y$ and 0 otherwise. For example, if $V = \{1, 2, 3, 4, 5\}$, $U = \{1, 3, 5\}$ then $C_U = 10101$, $C_\emptyset = 00000$, $C_V = 11111$. The total number of subsets of X (i.e. $|P(X)|$) is then equal to the total number of all possible binary strings of length n. There are two strings of length one (0 or 1), four strings of length two (00,01,10,11), eight strings of length three $\{000, 001, \ldots, 111\}$, etc., 2^n binary strings of length n.

Definition 3.5. The **ordered n-tuple** is the ordered collection $(a_1, a_2, \ldots, a_n)$ of elements. An important feature of ordered n-tuples is that

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \text{ yields } a_1 = b_1,\ a_2 = b_2, \ldots, a_n = b_n$$

In particular, when $n = 2$ we get the ordered **pairs.** Example: coordinates of points (x, y) on a plane is a typical ordered pair. Mind the difference between an ordered pair (a, b) and a not ordered pair $\{a, b\}$. $\{1, 2\} = \{2, 1\}$, but $(1, 2) \neq (2, 1)$, $\{1, 1\} = \{1\}$, but $(1, 1) \neq (1)$.

Definition 3.6. Cartesian product of two sets A and B is $A \times B = \{(a, b) \mid a \in A, b \in B\}$. The Cartesian product on n sets:

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Example: If $A = \{0, 1\}$, $B = \{a, b, c\}$, then $A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c)\}$.

Theorem 3.7. *If* $|A_1| = n_1,\ |A_2| = n_2, \ldots,\ |A_k| = n_k$, *then* $|A_1 \times A_2 \times \ldots \times A_k| = n_1 \cdot n_2 \cdot \ldots \cdot n_k$.

There is a remarkable correspondence between boolean logical connectives $\wedge, \vee, \neg$ and elementary operation on sets $\bigcap$ (**intersection**), $\bigcup$ (**union**) and $-$ (**complement**).

Definition 3.8. $A \bigcup B = \{x \mid x \in A \vee x \in B\}$, $A \bigcap B = \{x \mid x \in A \wedge x \in B\}$, $\overline{A} = \{x \mid x \notin A\} = \{x \mid \neg(x \in A)\}$. In the latter case the domain of variable x (the **universe**) should be specified exactly. Operations $\bigcap$ and $\bigcup$ admit the straightforward generalization to intersections and unions of arbitrary collection of sets (cf. the textbook).

Examples. The universe $\{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$. Then $A \bigcup B = \{1, 2, 3, 4\}$, $A \bigcap B = \{2, 3\}$, $\overline{A} = \{4, 5\}$. A neat way of representing set-theoretical operations is provided by so-called Venn Diagrams (cf. the slides and the textbook).

Proving sets equal. In order to prove that two sets U and V are equal if suffices to establish two inclusions: $U \subseteq V$ and $V \subseteq U$. For example, let us prove that $\overline{A \bigcup B} = \overline{A} \bigcap \overline{B}$.

First, we show that $\overline{A \bigcup B} \subseteq \overline{A} \bigcap \overline{B}$. Let $x \in \overline{A \bigcup B}$. Then $x \notin (A \bigcup B)$, i.e. $\neg(x \in (A \bigcup B))$, which yields $\neg(x \in A \vee x \in B)$. By the de Morgan Law, $x \notin A \wedge x \notin B$, therefore $x \in \overline{A} \wedge x \in \overline{B}$, i.e. $x \in (\overline{A} \bigcap \overline{B})$.

Second, we show the inverse inclusion $\overline{A} \bigcap \overline{B} \subseteq \overline{A \bigcup B}$: $x \in \overline{A} \bigcap \overline{B} \Rightarrow x \in \overline{A} \wedge x \in \overline{B} \Rightarrow$ $x \notin A \wedge x \notin B \Rightarrow \neg(x \in A \vee x \in B) \Rightarrow x \in \overline{A \bigcup B}$.

Using logical equivalences helps us to cut the length of the proof by half.

$$\overline{A \bigcup B} = \{x \mid x \in \overline{A \bigcup B}\} = \{x \mid \neg(x \in (A \bigcup B))\} = \{x \mid \neg(x \in A \vee x \in B)\} =$$
$$= \{x \mid (\neg x \in A) \wedge (\neg x \in B)\} = \{x \mid x \in \overline{A} \wedge x \in \overline{B}\} = \{x \mid x \in (\overline{A} \bigcap \overline{B})\} = \overline{A} \bigcap \overline{B}.$$

Homework assignments. (due Friday 02/02).

A. Section 1.4: 12

B. Section 1.5: 12de, 38.