CS280, Spring 2001 **Handout 29**
April 6, 2001

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 7.1, 7.2
2. The main message of this lecture:

Mathematical definition of all sorts of graphs.

In the simplest setting a graph is defined as a collection of vertices (i.e. a set) and a collection of edges (i.e. a set of pairs of vertices). However, we also have to provide exact mathematical definitions for other sorts of graphs: with directed edges, with multiple edges, with loops, etc.

Definition 29.1. A **simple graph** is a pair of sets $G = (V, E)$, where $V \neq \emptyset$ is called the set of **vertices**, and E a set of **edges** which are unordered pairs of distinct vertices. Two vertices u, v are called **adjacent** in G if there is an edge $\{u, v\}$ in G. The edge $e = \{u, v\}$ is said to **connect** u and v, which are called **endpoints** of this edge. We also say that $e = \{u, v\}$ is incident with u, v.

Simple graphs are usually visualized as a collection of nodes (vertices), some of them connected by lines (edges). Edges are not directed, each of them is nothing but an indication of a connection between two vertices. No edge goes from a node to itself. It is all right to have isolated nodes, i.e. vertices not connected.

Examples 29.2. The most common types of simple graphs (see pictures on the slides and in the book).

Complete graphs, denoted by K_n, $n \geq 1$: each pair of distinct vertices is an edge.

Cycles, denoted by C_n, $n \geq 3$: vertices $v_1, v_2, \ldots, v_n$, edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_n, v_1\}$.

Wheels W_n are obtained from cycles C_n ($n \geq 3$) by introducing a new $(n + 1)$st vertex and connecting it to all "old" vertices.

Cubes Q_n, $n \geq 1$, have 2^n vertices which are bit strings of length n. Two vertices are adjacent if they differ in exactly one bit position.

Definition 29.3. A simple graph is **bipartite** if its vertex set can by partitioned into two disjoint nonempty subsets such that each edge goes from one subset to another (there are no edges connecting vertices from the same subset).

Examples 29.4. Bipartite: K_2 - trivial

$C_4, C_6, C_8 \ldots$: partition vertices $v_1, v_2, \ldots, v_{2n}$ into subsets of odd and even ones

$Q_1, Q_2, Q_3, \ldots$: e.g. for Q_3 the partition is $\{000, 011, 101, 110\}$ and $\{001, 010, 100, 111\}$

Examples 29.5. Not bipartite: C_3, K_4, W_4, C_5

Examples 29.6. Complete bipartite graph $K_{m,n}$ is a graph which vertices are partitioned into two subsets of m and n vertices. Each pair of vertices from different subsets is an edge, no pair of vertices from the same subset are connected.

Examples 29.7. Computer network connected by telephone lines can be represented by a simple graph. Data flows both ways. No computer is connected to itself.

Examples 29.8. Local area networks: connects different computer devices. Possible connection schemes: a *star topology* - type $K_{1,n}$, a *ring topology* - type C_n, a *hybrid topology* - W_n.

Now we introduce some other types of graphs.

Definition 29.9. A **multigraph** admits multiple (undirected) edges between the same vertices (but no loops). A multigraph G is a triple (V, E, f), where $V \neq \emptyset$ is the set of vertices, E the set of edges, and f is a reading function from E to the set of pairs of distinct elements of V: for each $e \in E$ $f(e) = \{u, v\}$ where $u, v \in V$ and $u \neq v$. Standard examples: a computer network with multiple telephone lines, airlines connection maps.

Definition 29.10. A **pseudograph** is a multigraph with loops. A formal definition: a pseudograph is a triple (V, E, f), where $V \neq \emptyset$ is the set of vertices, E the set of edges, and f is a reading function from E to the set of pairs of elements of V, not necessarily distinct. Standard example: a computer network with multiple telephone lines, including self-connected for diagnostic purposes.

Definition 29.11. Directed graphs are graphs with directed edges. A **directed graph** $G = (V, E)$, where $V \neq \emptyset$ is the set of vertices, E is a relation on V (i.e. $E \subseteq V \times V$). Standard example: computer network with asymmetric lines, no multiple lines go in the same direction, loops are possible. A **directed multigraph** is a triple (V, E, f), where $V \neq \emptyset$ is the set of vertices, E the set of edges, and f is a reading function from E to V^2.

Definition 29.12. A **degree** *deg(v)* of a vertex $v \in V$ is the total number of edges incident with it. Note that each loop is counted twice.

Theorem 29.13. The Handshaking Theorem. *Let $G = (V, E)$ be an undirected (pseudo)graph (i.e. with possible loops and multiple edges), and let $e = |E|$. Then*

$$2e = \sum_{v \in V} deg(v)$$

Proof. Every edge is counted twice in Σ.

Corollary 29.14. *An undirected graph has an even number of vertices of odd degree.*
Proof. Let V_0 and V_1 be the sets of vertices of even degree and of odd degree respectively. By Theorem 29.13,

$$2e = \sum_{v \in V} deg(v) = \sum_{v \in V_0} deg(v) + \sum_{v \in V_1} deg(v)$$

Each term in the sum of degrees for $v \in V_0$ is even, therefore, the whole sum for $v \in V_0$ is even. Since the left hand side of this identity is also even, the remaining sum for $v \in V_1$ is even. The former consists of a certain number of *odd* terms (since only odd degrees are counted there). Therefore, the total number of those terms (i.e. the total number of vertices of odd degree) is even.

Definition 29.15. A subgraph of $G = (V, E)$ a given graph $H = (W, F)$ is a graph where both $V \subseteq W$ and $E \subseteq F$. Example: C_n is a subgraph of W_n with the natural correspondence between vertices.

Definition 29.16. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Example. C_3 has vertices v_1, v_2, v_3, $K_{1,3}$ has a vertex u in one part and the same v_1, v_2, v_3 in the other. Then $C_3 \cup K_{1,3} = W_3$.

Homework assignments. (The second installment due Friday 04/13)

29A:Rosen7.1-4,6,8; 29B:Rosen7.2-2; 29C:Rosen7.2-18; 29D:Rosen7.2-20; 29E:Rosen7.2-32.