CS280, Spring 2001 **April 4, 2001**	**Handout 28**

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 6.5, 6.6

2. The main message of this lecture:

Equivalence relations, partial orderings.

Definition 28.1. An **equivalence relation** on a set A is reflexive, symmetric and transitive (binary) relation on A. Examples: $a = b$, $a \equiv b\ (\mathbf{mod}\ m)$), *a is a relative of b*, etc.

Definition 28.2. Let R be an equivalence relation on A. For each $a \in A$ the **equivalence class** $[a]_R$ with respect to R is defined as the set of all elements of A that are R-equivalent to a: $[a]_R = \{b \mid (a,b) \in R\}$. Each such $b \in [a]_R$ is called a **representative** of the equivalence class $[a]_R$. Equivalence classes are all nonempty, since $a \in [a]_R$, by the reflexivity of R.

Examples: For the equality relation $[a]_= = \{a\}$, i.e. each equivalence class is a singleton
$[2]_{\equiv \mathbf{mod}\ 5} = \{\ldots, -13, -8, -3, 2, 7, 12, 17, \ldots\}$

Lemma 28.3. *For each equivalence relation any two equivalence classes are either disjoint or coincide:* $[a] \cap [b] \neq \emptyset \quad \Rightarrow \quad [a] = [b]$.
Proof. Suppose $[a] \cap [b] \neq \emptyset$ and take $c \in ([a] \cap [b])$. For such c we have both aRc and bRc. Since R is symmetric, cRb also holds. Since R is transitive, we conclude that aRb. It remains to show that aRb yields $[a] = [b]$. It suffices to check that $[a] \subseteq [b]$, since the second case $[b] \subseteq [a]$ has a similar justification. Let $d \in [a]$, i.e. dRa. Since aRb., by the transitivity of R, dRb, and therefore $d \in [b]$.

Definition 28.4. A **partition** of a set A is a collection of nonempty disjoint sets $\{A_i \mid i \in I\}$, such that

$$A = \bigcup_{i \in I} A_i.$$

Theorem 28.5. *Each equivalence relation R on A specifies a partition of A by the equivalence classes with respect to R. Conversely, given a partition $\{A_i \mid i \in I\}$ of A specifies an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.*
Proof. The equivalence classes with respect to R are disjoint, by Lemma 28.3. Moreover, they cover the whole of A, since every $a \in A$ belongs to it least one equivalence class $a \in [a]$. For part two consider an arbitrary partition $\{A_i \mid i \in I\}$ of A, and define the relation R by stipulating: $aRb \quad \Leftrightarrow \quad$ *a and b belong to the same partition set.* The relation R is obviously reflexive, and symmetric. Checking transitivity: let aRb and bRc. Then a and b are in some partition set A_i. Likewise, bRc yields b and c are in the same partition set A_i, therefore aRc.

Example 28.6. From an equivalence relation to a partition. Equivalence classes on the set of integers $\mathbf{Z}$ ($\mathbf{mod}$ 3).
$[0] = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$
$[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$
$[2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$ $\qquad \mathbf{Z} = [0] \cup [1] \cup [2]$.

Example 28.7. From a partition to an equivalence relation. On the same set of integers $\mathbf{Z}$ consider a partition $\mathbf{Z} = \{\ldots, -4, -3, -2, -1\} \cup \{0\} \cup \{1, 2, 3, 4, \ldots\}$. The equivalence relation

$\sim$ induced by this partition is $a \sim b \quad \Leftrightarrow \quad a$ *and* b *have the same sign*. This example illustrates that equivalence classes not necessarily have "the same" number of elements.

Definition 28.8. A (binary) relation R on A is called **partial ordering** (or **partial order**), if R is reflexive, antisymmetric, and transitive. The pair (A, R) is then called a **partially ordered set** (**poset**, for short). Traditionally, partial orderings are denoted $\preceq$, and used in the format $a \preceq b$.

Examples: $a \le b$ on integers, rationals, reals. $A \subseteq B$ on sets, $n|m$ on positive integers. In the first of those examples any two elements are related one way or another: $a \le b$ or $b \le a$. This does not hold, however, for examples two and three.

Definition 28.9. In a partially ordered set S two elements a, b are **comparable** if either $a \preceq b$ or $b \preceq a$. A partial order $(S, \preceq)$ is a **linear order** (or **total order**), if every two elements of S are comparable. A linear order $(S, \preceq)$ is **well-ordered**, if every nonempty subset X of S has the least element: $a \in X$ such that $a \preceq y$ for any $y \in X$.

Example 28.10. Linear orders: $\le$ on $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, lexicographic order on strings from $\mathbf{Z}$
Partial orders which are not linear: $\subseteq$ on sets, $(m, n) \preceq (m', n') \;\Leftrightarrow\; m \le m'$ *and* $n \le n'$
Well-orderings: $(\mathbf{Z}^+, \le)$, $\mathbf{Z}^+ \times \mathbf{Z}^+$ with lexicographic ordering
Linear orderings which are not well ordered: $(\mathbf{Z}, \le)$

Definition 28.11. Hasse diagrams for a poset S is a picture of S directed "upward" with all the redundant edges removed: loops, transitivity redundancies, arrows on edges (by arranging an initial vertex below its terminal vertex). Examples: see pictures from the textbook and from the slides.

Definition 28.12. Every ordering $\preceq$ has its natural irreflexive counterpart $\prec$ defined as $a \prec b \quad \Leftrightarrow \quad a \preceq b$ *and* $a \ne b$. An element $a \in S$ is a **maximal element** on a poset $(S, \preceq)$ if there is no $b \in S$ such that $a \prec b$. Likewise, $a \in S$ is a **minimal element** if there is no $b \in S$ such that $b \prec a$. An element $a \in S$ is the **greatest element** if $b \preceq a$ for all $b \in S$; $a \in S$ is the **least element** if $a \preceq b$ for all $b \in S$. It is immediate from the definitions that the greatest element (the least element) is unique, if any. It is also clear that the greatest element (the least element), if any, is a maximal (minimal) element.

Example 28.13. $(\mathbf{Z}, \le)$ has neither maximal or minimal elements, therefore, neither greatest nor least elements. $(\mathbf{N}, \le)$ has the least (therefore minimal) element 0, but no maximal elements. The set of all nontrivial subsets of $A = \{a, b, c\}$ (i.e. $X \subseteq A$ such that $X \ne \emptyset$ and $x \ne A$) ordered by inclusion $\subseteq$ has three distinct minimal elements $\{a\}$, $\{b\}$, and $\{c\}$, and three distinct maximal elements $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. The multiplicity of minimal (maximal) elements indicate that there are no greatest (least) elements there. Note that a minimal element is not necessarily the least element, even if this minimal one is unique!

Definition 28.14. Let $(S, \preceq)$ be a poset and $A \subseteq S$. An element $s \in S$ is an **upper bound** for A if $a \preceq s$ for all $a \in A$. Likewise, s is a **lower bound** for A if $s \preceq a$ for all $a \in A$. The **least upper bound** for A (notation $sup(A)$) is the least element, if any, among all upper bounds for A. Likewise the **greatest lower bound** for A (notation $inf(A)$) is the greatest among all lower bounds for A. Examples: $\mathbf{Q} \subset \mathbf{R}$ has neither upper nor lower bounds. The set $D = \{r \in Q \mid r^2 < 2\}$ does not have *sup* in $(\mathbf{Q}, \le)$, but has *sup* in $(\mathbf{R}, \le)$: $sup(D) = \sqrt{2}$.

Homework assignments. (The first installment due Friday 04/13) 28A:Rosen6.5-2; 28B:Rosen6.5-14b; 28C:Rosen6.5-22bd; 28D:Rosen6.6-2b; 28E:Rosen6.6-10ac; 28F:Rosen6.6-24.