1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 6.3, 6.4.
2. The main message of this lecture:

> **Theoretically speaking a relation is a set of ordered pairs (n-tuples). Speaking practically, a relation is a matrix, a graph, etc.**

As we remember, a (binary) relation $R \subseteq A \times B$ for finite A, B can be represented by a characteristic bit matrix $M_R = [m_{i,j}]$ where $m_{i,j} = 1$ if $(a_i, b_j) \in R$ and $m_{i,j} = 0$ if $(a_i, b_j) \notin R$. Therefore, the usual set theoretical operations on relations can be represented by bit operation on matrices.

$M_{R\cup S} = M_R \vee M_S$, (the *join* of matrices, the entrywise "$\vee$")
$M_{R\cap S} = M_R \wedge M_S$, (the *meet* of matrices, the entrywise "$\wedge$")
$M_{R\circ S} = M_S \odot M_R$, (the *boolean product* of matrices)

Mind the change of order of appearance of S and R in the last formula due to an awkward notation for $R \circ S$

Example 27.1. Let $A = \{a, b\}$ and R, S be relations on A represented by the matrices

$$M_R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad M_S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then

$$M_{R\cup S} = M_R \vee M_S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M_{R\cap S} = M_R \wedge M_S = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$M_{R\circ S} = M_S \odot M_R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Another canonical way of representing relations on A is **directed graphs** (or **digraphs**, for short). Elements of A are represented by **vertices** (or **nodes**) of a graph. Elements (a, b) of $R \subseteq A^2$ are represented by **edges** (or **arcs**) from a to b. To be absolutely honest, from the point of view of abstract mathematics, a relation and the representing digraph are the same objects, namely, sets of ordered pairs of vertices. The difference is in its visualizing: the former is usually thought of as a list of pairs, whereas the latter is a picture with nodes and arcs (see examples in the book and on the slides).

R is reflexive iff there is a loop as every node. R is symmetric iff for every edge there is the edge in the opposite direction. R is transitive iff for every two edges $a \longrightarrow b$ and $b \longrightarrow c$ there is a edge is $a \longrightarrow c$.

Definition 27.2. Let **P** be a property of relations (such as reflexivity, symmetry, transitivity). A closure of a given relation R with respect to **P** is the smallest relation S that contains R and has property **P**:

1. $S \supseteq R$
2. S has property P
3. $S \subseteq X$ for any X satisfying (1) and (2).

Example 27.3. Let R be a relation on A. The **reflexive closure** of R is $R \cup \{(a, a) \mid a \in A\}$. The symmetric closure of R is $R \cup R_1$, where $R_1 = \{(b, a) \mid (a, b) \in R\}$.

Definition 27.4. A **path** in a digraph G is a sequence of edges $(x_0x_1), (x_1x_2), \ldots, (x_{n-1}x_n)$ from G. Notation: a path $x_0x_1x_2 \ldots x_{n-1}x_n$, n is the length of a path = the number of edges (not nodes!). A path from a to b is a path $ax_1x_2 \ldots x_{n-1}b$, a **cycle** is a path $x_0x_1 \ldots x_{n-1}x_0$.

Theorem 27.5. *There is a path from a to b of length n in a digraph corresponding to a relation R if and only if $(a, b) \in R^n$.*

Proof. Induction on n. Base: $n = 1$. In this case a path is ab and $(a, b) \in R$. Induction Hypothesis: assume that the theorem holds for $n = k$. Step: there is a path from a to b of length $k + 1$ if and only if for some c such that $(c, b) \in R$ there is a path from a to c of length k. By the Induction Hypothesis, the latter is equivalent to $(a, c) \in R^k$, therefore the existence of a path from a to b of length $k + 1$ is equivalent to $(a, b) \in R^{k+1}$.

Definition 27.6. Let R be a relation on A. The **connectivity relation** over R is $R^* = \{(a, b) \mid \textit{ there is a path from } a \textit{ to } b \textit{ in } R\}$.

Corollary 27.7.

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

Theorem 27.8. *Let R be a relation on A. Then the transitive closure of R is R^*.*

Proof. *The transitive closure of* $R \subseteq R^*$ since $R^* \supseteq R$ and R^* is transitive. On the other hand, $R^* \subseteq$ *the transitive closure of* R since any connected pair (a, b) belongs to any transitive $S \supseteq R$.

Examples 27.9. The transitive closure of the relation on reals "the distance between x and y is one" is "the distance between x and y is an integer". The transitive closure of "a computer a has had a connection to a computer b" contains all pairs of computers from WWW. The transitive closure of "x is a mother of y" contains, in particular, the pair *(Eve, yourself)*.

Though generally speaking $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n \cup \ldots$, for *finite* relations the number of iterations in this union can be limited to the cardinality of the underlying set.

Theorem 27.10. *Let R be a relation on a finite set A. Then $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$ where $n = |A|$.*

Proof. Note that if there is a path from a to b in R then there is such a path of length not exceeding n. Indeed, if a path is longer then $n = |A|$, then, by the Pigeonhole Principle, this path contains cycles, that can be deleted (see the slides).

Corollary 27.11. Under the conditions of 27.10 $M_{R^*} = M_R \vee M_R^2 \vee M_R^3 \vee \ldots \vee M_R^n$

Homework assignments. (The third installment due Friday 04/06)

27A:Rosen6.3-8; 27B:Rosen6.3-12; 27C:Rosen6.4-20; 27D:Rosen6.4-26bc;