CS280, Spring 2001 **Handout 25**
March 28, 2001

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 5.5.
2. The main message of this lecture:

The Inclusion-Exclusion Principle admits an elegant generalization to n participating sets for each n.

Theorem 25.1. (a good old Inclusion-Exclusion Principle for two sets)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. Present $A \cup B$ as the union of three disjoint sets

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B).$$

The cardinality of the union of disjoint sets is the **sum** of their cardinalities, therefore

$$|A \cup B| = |A - B| + |B - A| + |A \cap B|.$$

On the other hand, $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$, and both of their unions are disjoint. Thus $|A| = |A - B| + |A \cap B|$, $|B| = |B - A| + |A \cap B|$, therefore $|A| + |B| = |A - B| + |B - A| + 2 \cdot |A \cap B|$. Thus

$$|A \cup B| = |A - B| + |B - A| + |A \cap B| = (|A - B| + |B - A| + 2 \cdot |A \cap B|) - |A \cap B| = |A| + |B| - |A \cap B|.$$

A "light" proof of the same fact: in the sum $|A| + |B|$ each element of $A - B$ and each element of $B - A$ has been counted once, whereas each element of $A \cap B$ has been counted **twice**: first in A, and then in B. To obtain an exact number $|A \cup B|$ one has to make an adjustments in $|A| + |B|$ by subtracting $|A \cap B|$.

What about three sets? How can one evaluate $|A \cup B \cup C|$ knowing cardinalities of A, B, C, their intersections, etc.? Look at the picture on slide 145, and use the "light" argument. Consider the sum $|A| + |B| + |C|$ and try to adjust this number to get $|A \cup B \cup C|$.

the "blue" elements (sets $A - (B \cup C)$, $B - (A \cup C)$, $C - (A \cup B)$) are counted once,
the "red" elements (sets $(A \cup B) - C$, $(B \cup C) - A$, $(A \cup C) - B$) are counted twice,
the "green" ones (set $A \cap B \cap C$) are counted three times.

Let us try this as an approximation to $|A \cup B \cup C|$:

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C|.$$

In this sum all "blue" and all "red" elements have been counted once (which is good), but the "green" ones have been counted 0 times! Indeed, if $x \in A \cap B \cap C$, then x has been counted in each of the six terms above: with "+" in $|A|$, $|B|$, and $|C|$, and with "−" in $|A \cap B|$, $|B \cap C|$, and $|A \cap C|$. To compensate this discrepancy we have to add $|A \cap B \cap C|$ back, with gives a beautiful formula of the Inclusion-Exclusion Principle for three sets:

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|.$$

Example 25.2. Every student in some class takes at least one of three courses: Math, Philosophy or Introduction to Wines.

The number M of Math students is 30	The number MP of Math and Phil students is 10
The number P of Phil students is 40	The number MW of Math and I.W. students is 20
The number W of I.W. students is 100	The number PW of Phil and I.W. students is 20

There are 5 students who take all three courses: Math, Phil, and I.W. (the number $MPVW$). How many students (N) are there in the class? By the Inclusion-Exclusion Principle,

$$N = M + P + W - MP - MW - PW + MPW = 30 + 40 + 100 - 10 - 20 - 20 + 5 = 125$$

Example 25.3. Similar setup, but now we know N the total number of students in the class (say, $N = 125$), as well as $M = 30$, $P = 40$, $W = 100$, $MP = 10$, $MW = 20$, $PW = 20$. How many students take all three courses (i.e. evaluate the number MPW)? By the Inclusion-Exclusion Principle, plot an equation

$$125 = 30 + 40 + 100 - 10 - 20 - 20 + X,$$

from which it immediately follows that $X = 5$.

Theorem 25.4. The Inclusion-Exclusion Principle in a general setting

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n| - \sum_{1 \le i < j \le n} |A_i \cap A_j| +$$

$$+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \quad \ldots \quad (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof. It suffices to establish that each element a of $A_1 \cup A_2 \cup \ldots \cup A_n$ has been counted in the right-hand side of this formula exactly once. Without loss of generality we assume that $a \in A_1 \cup A_2 \cup \ldots \cup A_r$ and $a \notin A_{r+1}$, $a \notin A_{r+2}, \ldots, a \notin A_n$. The number x of times the element a has been counted in the following groups of terms are:

$\|A_1\| + \|A_2\| + \ldots + \|A_n\|$	$r = C(r, 1)$ times,
$-\|A_1 \cap A_2\| - \|A_1 \cap A_3\| - \ldots$	$-C(r, 2)$ times,
$+\|A_1 \cap A_2 \cap A_3\| + \ldots$	$+C(r, 3)$ times,
....	
$(-1)^{r+1}(\|A_1 \cap A_2 \cap \ldots \cap A_r\| + \ldots)$	$(-1)^{r+1}C(r, r)$ times
....	
$(-1)^{n+1}\|A_1 \cap A_2 \cap \ldots \cap A_n\|$	0 times.

Therefore, $x = C(r, 1) - C(r, 2) + C(r, 3) - \ldots (-1)^{r+1}C(r, r)$. Since

$$C(r, 0) - C(r, 1) + C(r, 2) - C(r, 3) + \ldots (-1)^r C(r, r) = 0,$$

$C(r, 0) - x = 0$, thus $x = C(r, 0) = 1$.

Note that the Inclusion-Exclusion formula is not very practical for large n's, since the number of terms in the right-hand side of it equals to the number of nonempty subsets of $\{1, 2, 3, \ldots, n\}$ i.e. equals to $2^n - 1$.

Homework assignments. (The first installment due Friday 04/06)
25A:Rosen5.5-8; 25B:Rosen5.5-10; 25C:Rosen5.5-24.