CS280, Spring 2001	**Handout 24**
March 26, 2001	

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 5.1
2. The main message of this lecture:

Recursion applies to counting as well.

Definition 24.1. A **recurrence relation** R for a sequence $\{a_n\}$ is an equation (understood broadly) that expresses a_n in terms of some of the previous terms $a_0, a_1, \ldots, a_{n-1}$. A **solution** of a given recurrence relation R is a sequence $\{a_n\}$ of terms satisfying R.

Example 24.2. The size of a certain fish population in Cayuga lake can increase 10% a year due to natural growth. The harvesting rate is 1000 individuals per year. If the initial population size is 8000 individuals find the population size after 5 years.

Solution: a_n= the population size after n years.
$a_0 = 8000$ - the initial condition
$a_n = 1.1 \cdot a_{n-1} - 1000$ - the recurrence relation proper.

Note that there is no principal difference between an initial condition and a recurrence relation in the narrow sense. In particular, the problem above can be formally presented in the standard unified form 24.1:

$$a_n = \begin{cases} 8000, & \text{if } n = 0 \\ 1.1 \cdot a_{n-1} - 1000, & \text{if } n \geq 1 \end{cases}$$

Here $n_0 = 1$. The problem 24.2 has a unique solution:
$a_0 = 8000$
$a_1 = 1.1 \cdot a_0 - 1000 = 8800 - 1000 = 7800$
$a_2 = 1.1 \cdot a_1 - 1000 = 8580 - 1000 = 7580$
$a_3 = 1.1 \cdot a_2 - 1000 = 7338$
$a_4 = 1.1 \cdot a_3 - 1000 = 7072$
$a_5 = 1.1 \cdot a_4 - 1000 = 6779.2$
Can you explain the population size being a rational which is not an integer? Well, this is a difference between a real biological system (where the size of a fish population is always a nonnegative integer) and its mathematical model where this number is not necessarily integer.

Example 24.3. Some more familiar examples.

Recurrence relation	Solution
$a_n = a_{n-1} + d$	$a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (n-1) \cdot d, \ldots$ arithmetic progression
$a_n = a_{n-1} \cdot q$	$a_0, a_0 \cdot q, a_0 \cdot q^2, \ldots, a_0 \cdot q^{n-1}, \ldots$ geometric progression
$a_0 = 0, a_1 = 1, a_n = a_{n-2} + a_{n-1}$	$0, 1, 1, 2, 3, 5, 8, 13, \ldots$ Fibonacci numbers

Example 24.4. (Compound interest) The initial deposit is \$10000 at a bank yielding 5% per year with interest compounded annually. How much will be the amount after n years?

Recurrent equation is $S_0 = 10000,\ S_n = 1.05 \cdot S_{n-1}$. Solution sequence: $S_0 = 10000,\ S_1 = 1.05 \cdot S_0 = 1.05 \cdot 10\,000 = 10\,500\ \ldots\ S_{10} = 16\,288.95\ \ldots\ S_{30} = 315\,000\ \ldots\ S_{100} = 1\,315\,012.6$. So, everyone can become well off provided he/she lives long enough ...

Example 24.5. Find a recurrence relation for the number b_n of bit strings of length n that do not have two consecutive 0's: $b_0 = 1$ (only one null string), $b_1 = 2$ (two bit strings of length 1, both fit). Let now $n \geq 2$. We present $b_n = X + Y$, where X is the number of strings ending with 1 and Y the number of strings ending with 0. Note that $X = b_{n-1}$, since each such string ending with 1 is $x1$ where x is a string without two consecutive 0's. Moreover, $Y = b_{n-2}$, since each such string ending with 0 is $y10$, where y is a string without two consecutive 0's. The resulting equation is $b_0 = 1,\ b_1 = 2,\ b_n = b_{n-2} + b_{n-1}$ for $n \geq 2$. Solution: $1, 2, 3, 5, 8, 13, 21, \ldots$. In other words, $b_n = f_{n+2}$, where f_m is the mth Fibonacci number.

Example 24.6. Suppose a codeword is a string of decimal digits, a valid codeword is a codeword with even number of 0's. Let a_n stand for the number of valid codewords of length n. Then $a_0 = 1$ (the null string fits). Consider $n \geq 1$. Each valid codeword x of length n can be represented as $x = y\sigma$, where y is a codeword, and σ a decimal digit. There are two disjoint possibilities:

1) $\sigma \neq 0$ thus y is a valid codeword, 2) $\sigma = 0$ and y is an invalid codeword.

By the Product Rule, the number of variants (1) is $a_{n-1} \cdot 9$, the number of variants (2) is $10^{n-1} - a_{n-1}$. By the Sum Rule, $a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$. In particular, $a_1 = 8 \cdot 1 + 10^0 = 8 + 1 = 9$, which agrees with the direct observation that the valid codewords of length 1 are $1, 2, 3, \ldots, 9$.

Example 24.7. Messages are transmitted through a communication channel using two signals: one requires 1 microsecond, the other 2 microseconds. Find the total umber a_n of messages that can be sent in n microseconds (no blanks are permitted). Note that $a_0 = 1$, $a_1 = 1$. Let $n \geq 2$. Then each message x of length n falls into one of two disjoint classes:

1) $x = y\alpha$, where y is a message of length $n - 1$, α a short signal.

2) $x = z\beta$, where z is a message of length $n - 2$, β a long signal.

As before, $a_n = a_{n-1} + a_{n-2}$, therefore, $a_n = f_{n+1}$.

Example 24.8. Find a recurrence equation for the number C_n of ways to parenthesize the product of $n+1$ terms $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$. For example, there is only one way to parenthesize a "product" x_0, thus $C_0 = 1$. There is also only one way to parenthesize $x_0 \cdot x_1$, therefore, $C_1 = 1$. For C_2 consider a product $x_0 \cdot x_1 \cdot x_2$. There are two different ways to do it: $(x_0 \cdot x_1) \cdot x_2$ or $x_0 \cdot (x_1 \cdot x_2)$, thus $C_2 = 2$. For $n = 3$ we already have five possibilities:
$x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)),\ \ x_0 \cdot ((x_1 \cdot x_2) \cdot x_3),\ \ (x_0 \cdot x_1) \cdot (x_2 \cdot x_3),\ \ (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3,\ \ ((x_0 \cdot x_1) \cdot x_2) \cdot x_3$.
Here is a general argument: for a product $x_0 \cdot x_1 \cdot x_2 \cdot \ldots \cdot x_n$ first of all pick one of n multiplications as the outermost one. Each such pick breaks the problem of size n into two independent problems of the combined size $n - 1$. By the Sum Rule and the Product Rule,

$$C_n = C_0 \cdot C_{n-1} + C_1 \cdot C_{n-2} + \ldots + C_{n-1} \cdot C_0 = \sum_{k=0}^{n-1} C_k C_{n-k-1}.$$

The numbers C_n are called **Catalan numbers**; it can be shown that $C_n = C(2n, n)/(n+1)$.

Homework assignments. (due Friday 03/30).

24A:Rosen5.1-10; 24B:Rosen5.1-22; 24C:Rosen5.1-30