1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.7

2. The main message of this lecture:

> **Some optimization problems of modest size admit a brute force solution: generate all relevant combinations (permutations, etc.) and check them all. A key ingredient in such an approach is generating permutations, combinations, etc.**

Definition 23.1. Let A be a linearly ordered finite set. Without loss of generality we may assume that $A = \{1, 2, \ldots, n\}$ for some positive integer n. A **lexicographic ordering** "$\prec$" of the set of permutations (of not necessarily equal length) of A is defined by the following condition:

$a_1 a_2 \cdots a_k \prec b_1 b_2 \ldots b_r$ iff at the first j where a's and b's differ $a_j < b_j$. The empty symbol precedes all other symbols in A.

Example 23.2. Lexicographic ordering of all 3-permutations of $\{1, 2, 3\}$:

$$123 \prec 132 \prec 213 \prec 231 \prec 312 \prec 321$$

Lexicographic ordering of all possible permutations of $\{1, 2, 3\}$:

$$1 \prec 12 \prec 123 \prec 13 \prec 132 \prec 2 \prec 21 \prec 213 \prec 23 \prec 231 \prec 3 \prec 31 \prec 312 \prec 32 \prec 321$$

Example 23.3. Some pairs $a \prec b$ such that b is an immediate successor of a (among 5-permutations of $\{1, 2, 3, 4, 5\}$):

$$\begin{array}{ll} 12345 \prec 12354 & 12453 \prec 12534 \\ 12354 \prec 12435 & 21345 \prec 21354 \\ 12435 \prec 12453 & 54321 - \mathit{None} \end{array}$$

Here is a general method of **generating "the next" permutation**.

1. Given a permutation $a_1 a_2 \ldots a_n$ (for, example, 12453) find a_i that precedes a tail ordered in the reverse order

$$\ldots a_i < a_{i+1} > a_{i+2} > \ldots > a_{n-1} > a_n$$

(here it is $4 < 5 > 3$, i.e. $a_i = 4$). If there is no such a_i then the whole permutation is in the reverse order and it is the largest possible in the lexicographic ordering.

2. Find b_i = minimum of $a_{i+1}, a_{i+2} \ldots a_n$ which is greater than a_i (here $b_i = 5$). Put $b_{i+1}, b_{i+2}, \ldots, b_n$ to be the remaining of $\{a_i, a_{i+1} \ldots a_n\}$ in increasing order (here 34).

3. $\mathit{NEXT} = a_1 a_2 \ldots a_{i-1} b_i b_{i+1} \ldots b_n$ (here 12534).

Example 23.4. All 4-permutations in their lexicographic order generated by the above algorithm:

$$1234 \prec 1243 \prec 1324 \prec 1342 \prec 1423 \prec 1432 \prec 2134 \prec \ldots \prec 4213 \prec 4231 \prec 4312 \prec 4321$$

Example 23.5. The NEXT to 78312654 is 78314256.

Generating combinations is easier. Each combination X i.e. a subset $X \subseteq A = \{1, 2, \ldots, n\}$, can be identified with a bit string $\sigma_1\sigma_2\ldots\sigma_n$ where

$$\sigma_i = \begin{cases} 1, & \textit{if } i \in X \\ 0, & \textit{if } i \notin X \end{cases}$$

There is a natural order on binary strings: increasing binary expansions.

Example 23.6. The next largest string after 1001101011 is 1001101100.

To produce all binary expansions of length n, start with the string $\underbrace{000\ldots0}_{n}$. Then successively find the next largest expansion until the bit string $\underbrace{111\ldots1}_{n}$ is obtained.

Example 23.7. $n = 3$. As you see, the natural ordering on bit strings induces some peculiar ordering on subsets

000	001	010	011	100	101	110	111
$\emptyset$	$\{3\}$	$\{2\}$	$\{2,3\}$	$\{1\}$	$\{1,3\}$	$\{1,2\}$	$\{1,2,3\}$

Example 23.8. Find the next largest combination of $\{1, 2, 3, 4, 5\}$ for $X = \{2, 4, 5\}$. First, we convert X into a bit string $\sigma(X) = 01011$. Then we find the *next* bit string 01100, and convert it back to a subset: $Y = \{2, 3\}$. The resulting Y is the next largest after X.

Happy spring break. Have fun!

Homework assignments. (The second installment due Friday 03/30)
23A:Rosen4.7-4; 23B:Rosen4.7-6.