1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.6
2. The main message of this lecture:

More counting tools covering possible repetitions.

Theorem 22.1. *The number of r-permutations of n objects with repetitions is n^r.*
Proof. Here we are having r positions to fill, n variants to choose from for each of those positions. By the Product Rule, the total number of variants equals $\underbrace{n \cdot n \cdot \ldots \cdot n}_{r \text{ times}} = n^r$.

Example 22.2. 0-1 strings of length r: the total number is 2^r. The number of words of length 3 in English (case insensitive) is 26^3.

Example 22.3. Sampling with replacement. An urn contains 3 blue balls and 5 red ones. What is the probability of drawing 4 blues balls in a row if a ball is put back into the urn after it is drawn? Solution (not the optimal one): there are 8 balls total, 3 of them blue, in the urn. Assume the balls distinguishable. By 22.1, there are 8^4 possible outcomes (4-permutations with repetitions out of 8), 3^4 of them successful (4-permutations with repetitions out of 3 blue balls). Therefore, $p = 3^4/8^4 = (3/8)^4$. A bit more intelligent solution: each drawing is a Bernoulli trial with the probability of success 3/8. In four independent trials the probability of four successes is $(3/8)^4$.

Example 22.4. Combinations with repetitions. Imagine an unlimited supply of apples, oranges and pears. How many different servings of 4 fruits are there. All apples are indistinguishable (otherwise the number of different servings would be unlimited), the same holds for oranges and pears. Let us try the brute force first: 4A, 4O, 4P, 3A1O, 3A1P, 3O1A, 3O1P, 3P1A, 3P1O, 2A2O, 2A2P, 2O2P, 2A1O1P, 2O1A1P, 2P1A1O, the total number is 15. A general method: each combination can be represented by a string of four $*$'s and two $|$'s as shown below.

$$\underbrace{\ldots * \ldots}_{\sharp \text{ of apples}} \mid \underbrace{\ldots * \ldots}_{\sharp \text{ of oranges}} \mid \underbrace{\ldots * \ldots}_{\sharp \text{ of pears}}$$

For example, the combination 4A will be represented as $****||$, 2A1O1P as $**|*|*$. A string $*||***$ encodes the combination 1A3P. The total length of a string of $*$'s and $|$'s)here it is 6) is the sum of the length of a combination (here it is 4) and the number of types of objects to choose from minus one (here it is $3-1=2$). Each combination is totally determined when the exact locations of $|$'s (or, equivalently, the location of $*$'s) is fixed. Here it is $C(4+2,2) = C(6,2) = 6!/(4!\ 2!) = (6 \cdot 5)/2 = 15$.

Theorem 22.5. *There are $C(n+r-1,r)$ r-combinations from n elements with repetitions.*
Proof. Each such combination is represented by a string of r stars and $n-1$ bars. There are $C(n-1+r,r)$ such strings.

Example 22.6. What is a total number of possible combinations of five bills out of \$1, \$5, \$10, \$20, \$50, and \$100? Apply the formula from 22.5 for the number of 5-combinations with repetitions (the length of a sample) from 6 elements (the number of types of bills at our disposal). $C(6+5-1,5) = C(10,5) = 10!/(5!\ 5!) = 252$.

Example 22.7. Find the total number of solutions of $x_1 + x_2 + x_3 = 100$ in nonnegative integers x_1, x_2, x_3. Surprisingly enough, this number theoretical question is governed by the same scheme of combinations with repetitions. Indeed, think of x_1 as a number of apples, x_2 a number of oranges and x_3 as a number of pears out of total 100. Each solution of this equation can be represented by a string of 100 $*$'s and two $|$'s:

$$\underbrace{\ldots * \ldots}_{x_1} | \underbrace{\ldots * \ldots}_{x_2} | \underbrace{\ldots * \ldots}_{x_3}$$

Therefore, we apply the formula from 22.5 for the number of 100-combinations of 3 elements with repetitions ($n = 3$, $r = 100$): $C(3 + 100 - 1, 100) = C(102, 100) = C(102, 2) = (102 \cdot 101)/2 = 5151$.

Example 22.8. What is the total number of different permutations of the letters of the word *ILLINOIS*? If all the letters in this word were distinct then the total number of permutations would be 8!. In our situation, however, there are three indistinguishable *I*'s and two indistinguishable *L*'s. Therefore, real permutation corresponds to $2! \cdot 3! = 12$ distinguishable permutations. For example, the original string *ILLINOIS* is counted 12 times as $I_1L_1L_2I_2NOI_3S$, $I_1L_2L_1I_2NOI_3S$, $I_1L_1L_2I_3NOI_3S$, etc. The final count is $8!/(2!\ 3!) = 3360$.

Theorem 22.9. *The number of different permutations on n objects with $n_1, n_2, \ldots, n_k$ indistinguishable objects is*

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots \cdot n_k!}$$

Proof. If all objects were distinct then we'd have $n!$ permutations. Since n_i objects of type i are indistinguishable we counted each permutation $n_i!$ times for each $i = 1, 2, \ldots, k$. The total count is $n!/(n_1! \cdot n_2! \cdot \ldots \cdot n_k!)$.

Example 22.10. Distributing objects into boxes. Find the total number of hands of 5 cards to each of 4 players from the standard deck of 52 cards. Note that all 52 cards are distinct here. Hand one has $C(52, 5)$ possible variants. Hand two should be considered under the condition that hand one is chosen: $C(47, 5)$. Hand three (provided hands one and two are chosen) has $C(42, 5)$ variants. Finally, hand four (provided hands one, two and three are chosen) has $C(37, 5)$ variants. By the Product Rule, the total number of four hands is

$$C(52, 5) \cdot C(47, 5) \cdot C(42, 5) \cdot C(37, 5) = \frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}$$

The second solution: count the number of 52-permutations with $5, 5, 5, 5, 32$ indistinguishable objects. Indeed, we can start counting not from the deck of 52 cards, but from 52 imaginary positions: 5 in each of the four hands, and 32 remaining ones: $(I, 1)$, $(I, 2), \ldots, (IV, 5)$, $(R, 1), \ldots, (R, 32)$. Each distribution of hands may be regarded as an assignment of one of 52 pairs above to 52 individual cards, i.e. the number of 52-permutations of 52 elements with $5, 5, 5, 5, 32$ indistinguishable objects

$$\frac{52!}{5! \cdot 5! \cdot 5! \cdot 5! \cdot 32!}$$

Theorem 22.11. *The number of ways to distribute n distinct objects into k distinct boxes $n_1, n_2, \ldots, n_k$ objects each is*

$$\frac{n!}{n_1! \cdot n_2! \cdot \ldots ! \cdot n_k!}$$

Homework assignments. (The first installment due Friday 03/30)
22A:Rosen4.6-10ac; 22B:Rosen4.6-16ad; 22C:Rosen4.6-34; 22D:Rosen4.6-38.