1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.5
2. The main message of this lecture:

> **Expected value is the exact equivalent of an "average value" intuition.**

Example 21.1. To enter a game one has to pay \$3. Then a die is rolled, and the player gets paid the number of \$\$ equal to the number of points which the die comes up. Should a player accept the conditions and play this game? Imagine the game has been played 600 times (to get enough statistics). In about 1/6th of the cases (total about 100) the die comes up 1 and then the player loses $1 - 3 = -2$, the total loss from those will be $100 \cdot (-2)$. Similarly, in 1/6th of the cases the die comes up 2 and the whole loss is about $100 \cdot (-1)$. When the die shows 3, the player is even. The contributions of the cases when the die comes up 4, 5 or 6 are $100 \cdot 1$ $100 \cdot 2$ and $100 \cdot 3$ respectively. The estimate G of the overall gain/loss is then

$$G = 100 \cdot (-2) + 100 \cdot (-1) + 100 \cdot 0 + 100 \cdot 1 + 100 \cdot 2 + 100 \cdot 3 = 300$$

To get an estimate of the average gain/loss *per game* take

$$E = G/600 = \frac{100}{600} \cdot (-2) + \frac{100}{600} \cdot (-1) + \frac{100}{600} \cdot 0 + \frac{100}{600} \cdot 1 + \frac{100}{600} \cdot 2 + 100 \cdot 3 =$$

$$= \frac{1}{6} \cdot (-2) + \frac{1}{6} \cdot (-1) + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 = \frac{3}{6} = 1/2$$

The standard reading of this figure: as average, a player gains \$0.5 per game. Note that the last sum is nothing but

$$\sum_{i=1}^{6} \textit{(probability of i)} \cdot \textit{(the gain when i comes up)}$$

Example 21.2. A lottery ticket costs \$10, a probability to win is 1/5000000 and the winner gets \$10000000. How much a player wins/loses in average? We use the same idea:

$$E = \frac{1}{5000000} \cdot (10000000 - 10) + (1 - \frac{1}{5000000}) \cdot (-10) = 2 - \frac{10}{5000000} - 10 + \frac{10}{5000000} = -8\$.$$

The sign "–" indicates that a player loses the average of \$8 each game.

Definition 21.3.
A **random variable** $X(t)$ is a function from a sample space S to the set or reals $\mathbf{R}$.

Example 21.4. The cost function $X(t)$ from the game 21.1 may be regarded as a random variable that maps individual outcomes from the samples space $u_1, u_2, u_3, u_4, u_5, u_6$ to the real numbers $-2, -1, 0, 1, 2, 3$ respectively. In 21.2 the sample space $S = \{W, L\}$ has probabilities $p(W) = 1/5000000$, $p(L) = 4999999/5000000$. The function $X(W) = 10000000 - 10$, $X(L) = -10$ is a random variable.

Note that a random variable is neither random nor variable!

Definition 21.5. An **expected value** (or **expectation**) of a random variable $X(t)$ is

$$E(X) = \sum_{s\in S} p(s)\cdot X(s)$$

Example 21.6. $X(s)$ is the number of Heads out of three flippings of a coin. $X(TTT) = 0$, $X(HTT)=X(THT)=X(TTH)=1$, $X(HHT)=X(HTH)=X(THH)=2$, $X(HHH)=3$.

$$E(X) = \frac{1}{8}\cdot 0 + \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 1 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 3 =$$

$$= \frac{1}{8}\cdot 0 + 3\cdot\frac{1}{8}\cdot 1 + 3\cdot\frac{1}{8}\cdot 2 + \frac{1}{8}\cdot 1 = \frac{3}{2}$$

A grouping in a formula for expected value above can be generalized to a practical formula for computing the expected value:

$$E(X) = \sum_{r\in X(S)} p(X=r)\cdot r$$

Example 21.7. $X(s) =$ the sum on a pair of dice: $X(1,1) = 2, \ldots, X(6,6) = 12$. Then

$$E(X) = 2\cdot p(X=2) + 3\cdot p(X=3) + \ldots + 11\cdot p(X=11) + 12\cdot p(X=12) =$$

$$= 2\cdot\frac{1}{36} + 3\cdot\frac{2}{36} + 4\cdot\frac{3}{36} + 5\cdot\frac{3}{36} + 6\cdot\frac{5}{36} + 7\cdot\frac{6}{36} + 8\cdot\frac{5}{36} + 9\cdot\frac{4}{36} + 10\cdot\frac{3}{36} + 11\cdot\frac{2}{36} + 12\cdot\frac{1}{36} = 7$$

Theorem 21.8. *For X, Y random variables on a space S*
$E(X+Y) = E(X) + E(Y)$, $E(aX) = aE(X)$, $E(aX+b) = aE(X) + b$.

Proof. $E(X+Y) = \sum p(s)[X(s) + Y(s)] = \sum p(s)X(s) + \sum p(s)Y(s) = E(X) + E(Y)$
$E(aX) = \sum p(s)aX(s) = a\sum p(s)X(s) = aE(X)$, $E(b) = \sum p(s)\cdot b = b\sum p(s) = b\cdot 1 = b$,
$E(aX+b) = E(aX) + E(b) = aE(X) + b$.

Definition 21.9. Random variables X and Y are **independent** if
$p(X = a \text{ and } Y = b) = p(X=a)\cdot p(Y=b)$ for any $a\in X(S)$, $b\in Y(S)$.

Examples. Two dice, $X(i,j) = i$, $Y(i,j) = j$. Then $p(X = i \text{ and } X = j) = p(i,j) = 1/36$. On the other hand, $p(X=i) = p(Y=j) = 1/6$, therefore $p(X=i)\cdot p(Y=j) = 1/36$. Within the same model, X and $Z = X+Y$ are not independent, since $p(X = 1 \text{ and } Y = 10) = 0$ whereas $p(X=1)$ and $p(Z) = 1/12$.

Theorem 21.10. *If X, Y are independent then $E(X\cdot Y) = E(X)\cdot E(Y)$.*

Definition 21.11. The **variance** of X:

$$V(X) = E(X - E(X))^2 = \sum_{s\in S}[X(s) - E(X)]^2\cdot p(s)$$

The **standard deviation** $\sigma(X) = \sqrt{V(X)}$. Note that $V(X) = E(X - E(X))^2 =$
$= [X^2 - 2X\cdot E(X) + [E(X)]^2] = E(X^2) - E(2X\cdot E(X)) + [E(X)]^2 =$
$= E(X^2) - 2\cdot E(X)\cdot E(X) + [E(X)]^2 = E(X^2) - [E(X)]^2$.

Theorem 21.12. *If $X_1, X_2, \ldots, X_n$ are pairwise independent, then*
$V(X_1 + \ldots + X_n) = V(X_1) + \ldots + V(X_n)$.

Example: two dice $X(i,j) = i$, $Y(i,j) = j$ are independent, therefore, $V(X+Y) = V(X) + V(Y)$. $V(X) = E(X^2) - [E(X)]^2 = (1^2 + 2^2 + \ldots + 6^2)\cdot(1/6) - [(1+2+\ldots+6)\cdot(1/6)]^2 = 35/12 = V(Y)$. Therefore $V(X+Y) = 35/12 + 35/12 = 35/6$

Homework assignments. (due Friday 03/16)
21A:Rosen4.5-30; 21B:Rosen4.5-32; 21C:Rosen4.5-34; 21D:Rosen4.5-44.