1. Reading for Lecture 2: K. Rosen *Discrete Mathematics and Its Applications*, 1.3
2. The main message of the first lecture:

> **The whole of mathematics and some portions of natural language can be adequately represented using predicates and quantifiers** *for all* **and** *there exists*. **An important technical trick:** *individual variables*, **which were not explicitly present in the natural language.**

The logic of propositions is decidable, but has a very limited expressive power. In particular, it cannot emulate even the most basic properties of addition and multiplication on integers. Such syntactic constructions as *everybody loves somebody, nobody loves a looser, there is the least natural number but there is no the least rational,* contain references to properties common to all elements of very large or even infinite domain (all persons, all natural numbers, all rationals). An elegant formalization of such properties comes from mathematics and is based on notions of *individual variables* and *predicates*.

Definition 2.1. An **individual variable** is a letter (usually $x, y, z, \ldots$, possibly with indices) together with its **domain** (or the **universe of disclosure**), which is just some nonempty set. A **predicate** is a property of some number of objects of the domain. For example, let x be a variable over integers (the domain!), and $P(x)$ denote the predicate *"x is even"*. Other predicates over integers: $x \geq 3$, *"x is a whole square", "x is a prime", etc.* Predicates may depend on more than one variables: $x < y$, *"x loves y"*, *" x is a parent of y"*, $x+y=z$, *program p returns y in an input x*, etc. A predicate $P(x)$ may be regarded as *propositional function* depending upon x: for each specific element x of the domain $P(x)$ becomes a proposition, i.e. something which is either true or false. For example, if $P(x)$ is *"x is even"*, then $P(2)$ is true, but $P(3)$ is false.

Definition 2.2. Let $P(x)$ be a predicate. The **universal quantification** of $P(x)$ is a proposition $\forall x P(x)$ meaning *P(x) holds for all values from the domain of x.* Examples (here x, y, z are nonnegative integers):

$\forall x(x \textit{ is even})$ (false) $\quad \forall x(x > 2)$ (false)
$\forall x \forall y(x \leq y \vee y \leq x)$ (true) $\quad \forall x \forall y \forall z((x \leq y \wedge y \leq z) \rightarrow x \leq z)$ (true)

Example 2.3. Translating a natural language sentence into a formula with quantifiers. *Every student in this room studies discrete mathematics.* Introduce variables (English does not normally have them!) and predicates $S(x)$: *x is a student*, $R(x)$: *x is in this room*, $DM(x)$: *x studies discrete mathematics.* Write down a formula: $\forall x((S(x) \wedge R(x)) \rightarrow DM(x))$. This is easy, but takes some practice.

Definition 2.4. The **existential quantification** of a predicate $P(x)$ is a proposition $\exists x P(x)$ meaning *P(x) holds for some value from the domain of x.* Examples (here x, y are variables over reals):

$\exists x(x < 2)$ (true) $\quad \exists x(x = x + 1)$ (false) $\quad \exists x \forall y(x \leq y)$ (false) $\quad \forall y \exists x(x \leq y)$ (true)

Example 2.5. More translating natural language sentences into formulas. Here $M(x)$ denotes *"x is a mathematician"*, $P(x)$ – *"x understands politics"*, $L(x, y)$ – *"x loves y"*.

Someone in this room does not study discrete math	$\exists x(R(x) \wedge \neg DM(x))$
No mathematician understands politics	$\forall x(M(x) \rightarrow \neg P(x))$
Everybody loves somebody	$\forall x \exists y L(x,y)$
Someone loves nobody	$\exists x \forall y \neg L(x,y)$

Duality of quantifiers (follows immediately from the definitions):

$$\neg \forall x P(x) \Leftrightarrow \exists x \neg P(x) \qquad \neg \exists x P(x) \Leftrightarrow \forall x \neg P(x)$$

Examples: *Not everyone studies math* $\Leftrightarrow$ *There is someone who does not study math*
Nobody is a fool $\Leftrightarrow$ *Everybody is not a fool.*

Negating quantifies sentences. $\neg \forall x \exists y P(x,y) \Leftrightarrow \exists x \neg \exists y P(x,y) \Leftrightarrow \exists x \forall y \neg P(x,y)$.
A rule: *change all the quantifiers to their duals and negate the core.*

Redundancies in quantifiers. $\forall$ can be expressed via $\exists$ and vice versa. Indeed,

$$\forall x P(x) \Leftrightarrow \neg \exists x \neg P(x) \qquad \exists x P(x) \Leftrightarrow \neg \forall x \neg P(x)$$

Free and bound variables. A question *whether* $x < y$ *is true* (x, y positive integers) does not have a definite answer, since it depends on specific values of x and y, which are **free** here, i.e. are not in the scope of any quantifier. Suppose we bind one of the variables by a quantifier, say $\exists x(x < y)$. The truth value is still not determined! The variable x is already **bound** by $\exists x$, but y remains free, and a truth value of a sentence depends on specific choice of y: $\exists x(x < 1)$ is false (here $y = 1$), but $\exists x(x < 2)$ is true ($y = 2$). Binding y by quantifiers leads to different truth values: $\forall y \exists x(x < y)$ is false, $\exists y \exists x(x < y)$ is true. Bound variables can be safely renamed by a fresh variable:

$$\forall x \exists y A(x,y) \quad \Leftrightarrow \quad \forall u \exists v A(u,v).$$

Finite domains. For a finite domain $D = \{d_1, d_2, \ldots, d_n\}$ a formula $\forall x A(x)$ is equivalent to $A(d_1) \wedge A(d_2) \wedge \ldots \wedge A(d_n)$ and $\exists x(A(x)$ is equivalent to $A(d_1) \vee A(d_2) \vee \ldots \vee A(d_n)$. These indicates that $\forall$ may be regarded as a sort of a (possibly infinite) conjunction whereas $\exists$ is a sort of a (possibly infinite) disjunction.

Distributing quantifiers through boolean connectives. The following equivalences follow immediately from definitions:

$\forall x(A(x) \wedge B(x)) \Leftrightarrow \forall x A(x) \wedge \forall x B(x)$ ($\forall$ commutes with $\wedge$)
$\exists x(A(x) \vee B(x)) \Leftrightarrow \exists x A(x) \vee \exists x B(x)$ ($\exists$ commutes with $\vee$).

Note that generally speaking neither $\forall x(A(x) \vee B(x)) \Leftrightarrow \forall x A(x) \vee \forall x B(x)$, nor $\exists x(A(x) \wedge B(x)) \Leftrightarrow \exists x A(x) \wedge \exists x B(x)$. Indeed, let $A(x)$ be *"x is even"* and $B(x)$ be *"x is odd"* for integers. Then $\forall x(A(x) \vee B(x))$ holds, but not $\forall x A(x) \vee \forall x B(x)$. Likewise, $\exists x A(x) \wedge \exists x B(x)$ is true, but $\exists x(A(x) \wedge B(x))$ is false.

Quantifiers and implication. Let C do not depend on x. Then

$$\forall x(C \rightarrow A(x)) \Leftrightarrow C \rightarrow \forall x A(x) \qquad \forall x(A(x) \rightarrow C) \Leftrightarrow \exists x A(x) \rightarrow C$$
$$\exists x(C \rightarrow A(x)) \Leftrightarrow C \rightarrow \exists x A(x) \qquad \exists x(A(x) \rightarrow C) \Leftrightarrow \forall x A(x) \rightarrow C$$

Homework assignments. (due Friday 02/02).

A. 1.3 6,20

B. Rewrite $\neg \forall x \exists y(A(x,y) \rightarrow B(x,y))$ so that no negation appears outside quantifier or an expression involving logical connectives (i.e. move $\neg$ inside as much as possible).

C. Show that $\forall x(A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$ holds for all predicates $A(x)$ and $B(x)$. Show that the inverse implication does not necessarily hold, i.e. give an example of specific predicates $A(x)$ and $B(x)$ such that $\forall x A(x) \rightarrow \forall x B(x)$ is true but $\forall x(A(x) \rightarrow B(x))$ is false.