1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.4
2. The main message of this lecture:

> **The classical definition of probability came from the gambling games theory: the ratio of a number of successful outcomes to the of number of all possible outcomes, presuming they are all equally likely.**

Imagine a certain experiment that can equally likely produce some known finite number n of outcomes, for example, a die is rolled, $n = 6$. Imagine also that we bet on some kind of outcomes, for example, that a die comes up an even number, here 2, 4, or 6. Intuitively, the probability P of us winning is the number of k successful outcomes divided by the total number of outcomes n:

$$P = \frac{k}{n}$$

Definition 19.1. Experiment is a procedure that yields one of a given set of possible outcomes. **Sample space** S is the set of all possible outcomes of a experiment. **Event** E is a subset of the sample space. The probability of an event E is

$$p(E) = \frac{|E|}{|S|}.$$

It follows immediately from the definition that $0 \leq p(E) \leq 1$. To establish other fundamental properties of probability we will need to know counting. Note that there are two major assumptions under which this classical definition of probability operates:

1. Sample space S is finite, 2. All outcomes in S are equally likely.

Those conditions are often met, but in many applications one has to consider infinite state spaces with pretty sophisticated probability distribution on them. In general, probability is a very hard area of mathematics.

Example 19.2. Experiment: two dice are rolled. The sample space is the product $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} = \{(1,1)(1,2)\dots(1,6)(2,1)(2,2)\dots(2,6)(3,1)\dots(6,6)\}$. Possible outcomes are ordered pairs (*the first roll, the second roll*). The total number of possible outcomes is $|S| = 6 \cdot 6 = 36$. An event: *the sum of the two dice is five* gives the set $e = \{(1,4)(2,3)(3,2)(4,1)\}$, four pairs total. The probability that when two dice are rolled the sum is five is $p = 4/36 = 1/9$.

Example 19.3. An urn contains three blue balls and five red balls. What is the probability that a randomly chosen ball is blue? Imagine that the balls are numbered. Then S consists of 8 possible outcomes of picking an individual ball, $|S| = 8$. Those outcomes are assumed equally likely (the code word for this assumption is "randomly"). Since any of three blue balls does it, the event E here consists of three successful outcomes, $|E| = 3$. The probability $p = 3/8$.

Example 19.4. (Lottery) What is the probability to pick the correct six numbers out of 40? Sample space S here is the set of all possible 6-combinations out of 40.

$$|S| = \frac{40!}{6!\,34!} = 3838380$$

Event E here contains only one winning combination, $|E| = 1$. The probability of winning is then $p = 1/3838380$.

Example 19.5. (Poker) Find the probability that a hand of five cards contains four cards of the same kind. Here S is the set of all possible hands of five cards. There are 13 different kinds, four cards each, the total $13 \cdot 4 = 52$ cards in a deck. Since a hand is not ordered, we use combinations to evaluate the number of hands: $|S| = C(52, 5)$. E is the set of all hands containing four cards of the same kind. To evaluate $|E|$ we use specify a hand from E by stages and use the Product Rule.

Stage 1: picking a kind for four cards. There are 13 kinds, $C(13, 1)$ choices. After a kind is selected we have four cards out of five in a hand chosen.

Stage 2: picking a fifth card out of remaining $52 - 4 = 48$ can be made in $C(48, 1)$ ways. The probability of getting four card of one kind in an a hand is then

$$p = \frac{C(13,1) \cdot C(48,1)}{C(52,5)} = \frac{13 \cdot 48}{2598960} \sim 0.00024$$

Definition 19.6. Complementary event $\overline{E} = S - E$. Obviously, $|\overline{E}| = |S| - |E|$, therefore

$$p(\overline{E}) = \frac{|\overline{E}|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - p(E). \text{ Likewise, } \; p(E) = 1 - p(\overline{E})$$

Example 19.7. Find the probability that an integer x ($0 \leq x \leq 999999$) has at least one digit 8 in its decimal expansion. We assume tacitly that each of 10 integers $0, 1, 2, \ldots, 9$ is equally likely to appear in each of 6 decimal positions. S = total number of integers from 0 to 999999, $|S| = 10^6$. E is the set of those numbers from S which contain at least one digit 8. There is a straightforward tedious way of evaluating $|E|$: consider cases when the number of 8s is one, two, three, etc, six. Choose positions for those 8s, fill the remaining positions with digits other then 8, etc. If you have time and do not make many mistakes in calculations, you could eventually come with the right answer. However, considering the complementary event and then using a formula from 19.6 makes life much easier here. $\overline{E}$ consists of all numbers *not containing* 8s, $|E| = 9^6$ (there are six positions to fill and nine digits $\{0, 1, \ldots, 7, 9\}$ to choose from independently for each position). Therefore, $p(\overline{E}) = 9^6/10^6 = (0.9)^6 \sim 0.53$, $p(E) = 1 - p(\overline{E}) \sim 1 - 0.53 = 0.47$.

Example 19.8. (Tossing a coin) A sequence of 10 bits is randomly generated. What is the probability that at least one of them is 1? E is "at least one bit is 1", $\overline{E}$ is "all bits are 0". It is clear, that counting $\overline{E}$ is much easier: $|\overline{E}| = 1$, and $p(\overline{E}) = 1/2^{10} = 1/1024$. Then $p(E) = 1 - p(\overline{E}) = 1 - 1/1024 = 1023/1024$.

Theorem 19.9. $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
Proof. Since $|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$, $p(E_1 \cup E_2) =$

$$= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Example 19.10. A positive integer ≤ 100 is randomly selected. What is the probability that it is divisible by 2 or 3? Let E_1 be "x is divisible by 2", $p(E_1) = 50/100$. Let E_2 be "x is divisible by 3", $p(E_2) = 33/100$. Then $E_1 \cap E_2$ is x is divisible by both 2 and 3", $p(E_1 \cap E_2) = 16/100$. We have to evaluate $p(E_1 \cup E_2)$, which is then equal to $50/100 + 33/100 - 16/100 = 67/100$.

Homework assignments. (due Friday 03/16).

19A:Rosen4.4-6; 19B:Rosen4.4-16; 19B:Rosen4.4-28; 19c:Rosen4.4-32.