CS280, Spring 2001	Handout 18
March 2, 2001	

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.3
2. The main message of this lecture:

> **Permutations, combinations, binomial coefficients are all applications of the Product Rule of counting.**

Definition 18.1. A k-**permutation** is an ordered k-tuple of distinct objects.

Example 18.2. $S = \{a, b, c\}$. All 3-permutations: (a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a). All 2-permutations: (a, b), (a, c), (b, a), (b, c), (c, a), (c, b).

Theorem 18.3. *The total number of r-permutations of a set with n distinct elements is* $P(n, r) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) = n!/(n-r)!$.
Proof. There are n ways of choosing the first element of an r-permutation, $n-1$ ways for the second element out of $n-1$ remaining, $n-2$ ways for the third, etc. By the Product Rule, the total number of variants is the product $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1)$.

Example 18.4. The number of ways to award three (gold, silver and bronze) medals to ten players is $P(10, 3) = 10 \cdot 9 \cdot 8 = 720$. Indeed, each variant is a 3-permutation of players.

Corollary 18.5. *The number of permutations of an n-element set is $P(n, n) = n!$.*

Definition 18.6. An r-**combination** of a set S is an r-element subset of S. The total number of r-combinations of n-element set is denoted by $C(n, r)$, also called a **binomial coefficient**. Alternative notations: $\binom{n}{r}$, C_n^r.

Example 18.7. Mind the difference between r-permutations and r-combinations: the former are *ordered* r-tuples, whereas the latter are *not ordered* sets of distinct elements. Therefore there are less combinations than permutations. For example, consider the same $S = \{a, b, c\}$. There are only three 2-combinations $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ (versus six 2-permutations). In fact, one 2-combination (e.g. $\{b, c\}$ corresponds to two 2-permutations (b, c) and (c, b).

Theorem 18.8.

$$C(n, r) = \frac{n!}{r!\ (n-r)!}$$

Proof. Each r-combination (as an r-element set) generates $P(r, r)$ permutations of length r (i.e. r-permutation). Therefore, the total number of r-permutations equals to the total number of r-combinations taken $P(r, r)$ times: $P(n, r) = C(n, r) \cdot P(r, r)$. Hence

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!\ r!}$$

Example 18.9. The number of ways to pick a team of three hackers out of 20 to represent Cornell at a competition in Europe is

$$C(20, 3) = \frac{20!}{3!\ (20-3)!} = \frac{20!}{3!\ 17!} = \frac{20 \cdot 19 \cdot 18}{3!} = 20 \cdot 19 \cdot 3 = 1140$$

There is a trivial identity about binomial coefficients: $C(n,r) = C(n,n-r)$. Indeed,

$$C(n,n-r) = \frac{n!}{(n-r)!\ (n-(n-r))!} = \frac{n!}{(n-r)!\ (n-n+r))!} = \frac{n!}{(n-r)!\ r!} = C(n,r)$$

Theorem 18.10. (Pascal's Identity) $C(n+1,k) = C(n,k-1) + C(n,k)$.
Proof. Let T be a $n{+}1$ element set and $a \in T$. Put $S = T - \{a\}$; obviously, $|S| = n$. The number of k-combinations $X \subseteq T$ such that $a \in X$ equals to $C(n,k-1)$. Indeed, there is a one-to-one correspondence f between the set of such k-combinations and the set of $k-1$-combinations in S: $f(X) = X - \{a\}$. The number of k-combinations $X \subseteq T$ such that $a \notin X$ equals to the number of k-subsets of S, i.e. to $C(n,k)$. By the Sum Rule, $C(n{+}1,k) = C(n,k{-}1) {+} C(n,k)$.

Pascal Triangle is a table below, were the n-th row consists of the binomial coefficients $C(n,0) = 1$, $C(n,1) = n$, $\ldots$, $C(n,n-1) = n$, $C(n,n) = 1$. Pascal's Identity shows that an element in n-th row, $n \neq 1$, is a sum of two adjacent entries above it.

															n
							1								0
						1		1							1
					1		2		1						2
				1		3		3		1					3
			1		4		6		4		1				4
		1		5		10		10		5		1			5
	1		6		15		20		15		6		1		6
1		7		21		35		35		21		7		1	7

Theorem 18.11.

$$\sum_{r=0}^{n} C(n,r) = 2^n \qquad (\textit{in the alternative notation} \quad \sum_{r=0}^{n} \binom{n}{r} = 2^n).$$

Proof. The total number of subsets of an n-element set is 2^n. On the other hand this number is the sum of the numbers of r element subsets for r ranging from 0 to n, i.e.

$$2^n = C(n,0) + C(n,1) + C(n,2) + \ldots + C(n,n-2) + C(n,n-1) + C(n,n)$$

Example 18.12. $(x+y)^3 = (x+y)(x+y)(x+y) =$
$$= \underbrace{xxx}_{x^3} + \underbrace{xxy + xyx + yxx}_{x^2y} + \underbrace{xyy + yxy + yyx}_{xy^2} + \underbrace{yyy}_{y^3} = x^3 + 3x^2y + 3xy^2 + y^3.$$

Theorem 18.13. (The Binomial Theorem)

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Proof. $(x+y)^n = (x+y)(x+y)\ldots(x+y)$. To get $x^{n-j}y^j$ in the expansion we have a choice of $n-j$ terms $(x+y)$ out of n available to pick x from (the rest j terms $(x+y)$ would donate y's). The total number of $x^{n-j}y^j$'s in the expansion will then be $C(n,n-j)$.

Corollary 18.14. $C(n,0) - C(n,1) + C(n,2) - \ldots + (-1)^{n-1}C(n,n-1) + (-1)^n C(n,n) = 0$
Proof. In the binomial theorem put $x = 1$, $y = -1$.

Example 18.15. $(x-2y)^6 = x^6 - 6\cdot 2x^5y + 15\cdot 4x^4y^2 - 20\cdot 8x^3y^3 + 15\cdot 16x^2y^4 - 6\cdot 32xy^5 + 64y^6$.

Homework assignments. (due Friday 03/09).
18A:Rosen4.3-8; 18B:Rosen4.3-14; 18C:Rosen4.3-32; 18D:Rosen4.3-38.