CS280, Spring 2001	**Handout 17**
February 28, 2001	

1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.1, 4.2
2. The main message of this lecture:

> **Counting the number of elements in a finite set is pivotal in probability, complexity, and many other areas. Though there are basic methods to counting, it becomes tricky when a set is specified by some sophisticated condition.**

Example 17.1. Imagine there are three GOP and two Democratic Party candidates running for nominations from their parties. How many possibilities to choose the next president are there? (Assume that only a major party candidate has a chance to be elected).
Solution. Three variants from GOP, two variants from Dem, five variants altogether.

Theorem 17.2. (The Sum Rule) *If each solution of a task T is either a solution of T_1 (n_1 variants) or a solution of T_2 (n_2 variants) and there are no common solutions for these tasks, then the total number of solutions of T is $n_1 + n_2$.*
Proof. If $A \cap B = \emptyset$ then $|A \cup B| = |A| + |B|$.

Example 17.3. An assistant professor is looking for a computer for her office. There are 13 power PCs, 22 power Macs and 7 SUNs in a store. How many variants to pick a computer does she have? By the Sum Rule (twice), the number of variants is $13 + 22 + 7 = 42$.

Example 17.4. What is the value of k after the following code has been executed.

```
k := 0
for i := 1 to n_1
    k := k + 1
  ...
for i := 1 to n_m
    k := k + 1
```

The task of computing k has been split into subtasks $T_1, T_2, \ldots, T_m$ each having $n_1, n_2, \ldots, n_m$ "solutions". By the Sum Rule, $k = n_1 + n_2 + \ldots + n_m$. This example shows that the meaning of "tasks" and "solutions" should be understood in a very broad sense. In fact, the language of sets is the only honest (though pretty dry) language for counting.

Example 17.5. There are three GOP and two Dem candidates for presidential nominations. How many possibilities are there to have a contesting pair on the election day?
Solution. There are three choices for a GOP position in the final pair, say $\{G_1, G_2, G_3\}$. Independently, there are two choices for a Dem position, say $\{D_1, D_2\}$. Then there are exactly $3 \cdot 2 = 6$ possible variants for the final pair: $\{G_1, D_1\}$, $\{G_2, D_1\}$, …, $\{G_2, D_2\}$, $\{G_3, D_2\}$.

Theorem 17.6. (The Product Rule) *If each solution is an ordered pair (x, y) and there are n_1 choices for x and n_2 choices for y, then the total number of solutions is $n_1 \cdot n_2$.*
Proof. $|A \times B| = |A| \cdot |B|$.

Example 17.7. Each seat in a theater is labeled (L, N) where L is a letter and N is a positive integer, $N \leq 100$. How many labels are there?
Solution. By the Product Rule, $26 \cdot 100 = 2600$.

Example 17.8. What is the total number of bit strings of length ten?
Solution. There are ten positions, two choices for each. Using the Product Rule nine times we get the total number of variants (here – bit strings of length ten) $\underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{10 \ times} = 1024$.

Example 17.9. How many functions are there from $A = \{a, b, c\}$ to $B = \{0, 1\}$?
Solution. Each such function f can be represented by a triple of its values $f(a), f(b), f(c)$, each of which is either 0 or 1. By the multiple Product Rule, the total number of those triples is $2 \cdot 2 \cdot 2 = 8$.

Theorem 17.10. *If $|A| = m$ and $|B| = n$ then there are exactly n^m functions from A to B.*
Proof. Each function f is an m-tuple $f(a_1), f(a_2), \ldots, f(a_m)$, where there are n choices for each of the positions. By the multiple Product Rule, the total number of such strings is $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m \ times} = n^m$.

Example 17.11. Counting 1-1 functions from A to B ($|A| = m$, $|B| = n$). If $m > n$ there are none. Assume $m \le n$. A function is a string of length m. There are n choices for the first position. After it is filled, there are $n - 1$ choices for the second position, because one cannot use the same element as a value twice. Then there are $n - 2$ choices remaining for the third position, etc. Answer: $n \cdot (n - 1) \cdot \ldots \cdot (n - m + 1)$.

Example 17.12. A password is six to eight characters long. Each character is a digit, a lower case letter or an upper case letter. There should be at least one digit. What is the total number of passwords?
Solution. $P = P_6 + P_7 + P_8$, where P_i is the number of pw's of length i.

P_6= $\sharp$ *of strings of letters and digits* - $\sharp$ *of strings of letters only*= $62^6 - 52^6$,
since there are 26 lowercase letters, 26 upper case letters and 10 digits. Likewise, $P_7 = 62^7 - 52^7$, $P_8 = 62^8 - 52^8$.

Theorem 17.13. (The Inclusion-Exclusion Principle) $|A \cup B| = |A| + |B| - |A \cap B|$.
Proof. In $|A| + |B|$ each element of $|A \cap B|$ has been counted twice. To get the fair number of elements in $|A \cup B|$ we have to subtract $|A \cap B|$ from $|A| + |B|$.

Example 17.14. An educational committee has seven politicians and ten teachers. How many people are there in the committee if two of the politicians are also teachers.
Solution. By the Inclusion-Exclusion Principle, $|P \cup T| = |P| + |T| - |P \cap T| = 7 + 10 - 2 = 15$.

Theorem 17.15. (The Pigeonhole Principle) *If $\ge k + 1$ pigeons are places into k holes then there is a hole containing two or more pigeons.*

Example 17.16. Any group of 367 people has a pair with the same birthday.

Theorem 17.17. (The Generalized Pigeonhole Principle) *If N objects are places into k boxes then there is a box containing at least $\lceil N/k \rceil$ objects.*
Proof. Otherwise $N \le k(\lceil N/k \rceil - 1) < k(N/k + 1 - 1) = N$, a contradiction.

Example 17.18. Among 100 people there are at least $\lceil 100/12 \rceil = \lceil 8.333 \rceil = 9$ who were born in the same month.

Homework assignments. (due Friday 03/09).

17A:Rosen4.1-6; 17B:Rosen4.1-12; 17C:Rosen4.1-16; 17D:Rosen4.1-18acd; 17E:Rosen4.1-36; 17F:Rosen4.1-46; 17G:Rosen4.2-2; 17H:Rosen4.2-16; 17I:Rosen4.2-34b