1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 2.6
2. The main message of this lecture:

> **Matrices is a natural generalization of numbers. They have the same operations (addition/subtraction, multiplication/division), though with some peculiar properties: multiplication is not commutative, division is not always possible even when the denominator is not zero. Important: memorize how to multiply matrices!**

Numbers here are not necessarily integers.

Definition 11.1. Matrix is a rectangular array of $m \times n$ entries. Here m is the number of rows, n - the number of columns. Notations:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} = [a_{ij}] \text{ (i is the row index, j is the column index)}$$

Examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ (2×3 matrix)}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \text{ (3×2 matrix)}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ (4×1 matrix)}$$

Definition 11.2. Zero matrixes: for any given size $(m \times n)$ $\mathbf{0}$ is the matrix where all entries are equal to 0. **Addition** of matrices of the same size: add the corresponding entries $A+B = [a_{ij}+b_{ij}]$. It is clear that $A+\mathbf{0} = \mathbf{0}+A = A$. Examples: $(1\,2\,3\,4)+(5\,6\,7\,8) = (6\,8\,10\,12)$,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 11 & 12 & 13 \\ 14 & 15 & 16 \end{pmatrix} = \begin{pmatrix} 12 & 14 & 16 \\ 18 & 20 & 22 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

Definition 11.3. Multiplication of matrices A $(m \times n)$ and B $(n \times k)$. The product of A, B is $A \cdot B = C$ $(m \times k)$, where C is obtained by the *row-column rule*: to get c_{ij} multiply the corresponding entries from ith row of A and from $j$the column of B, and take the sum of the products.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj} = \sum_{l=1}^{n} a_{il}b_{lj}$$

Examples:

$$(1\ 2\ 3\ 4) \cdot \begin{pmatrix} -4 \\ -3 \\ -2 \\ -1 \end{pmatrix} = 1 \cdot (-4) + 2 \cdot (-3) + 3 \cdot (-2) + 4 \cdot (-1) = -20$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -6 & 5 \\ -4 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot(-6)+2\cdot(-4)+3\cdot(-2) & 1\cdot 5+2\cdot 3+3\cdot 1 \\ 4\cdot(-6)+5\cdot(-4)+6\cdot(-2) & 4\cdot 5+5\cdot 3+6\cdot 1 \\ 0\cdot(-6)+1\cdot(-4)+0\cdot(-2) & 0\cdot 5+1\cdot 3+0\cdot 1 \end{pmatrix} = \begin{pmatrix} -20 & 14 \\ -56 & 41 \\ -4 & 3 \end{pmatrix}$$

Matrix multiplication of A $(m \times n)$ and B $(n \times m)$ is not commutative, i.e., generally speaking, $A \cdot B \neq B \cdot A$. Indeed, even the sizes of $A \cdot B$ $(m \times m)$ and $B \cdot A$ $(n \times n)$ do not match. The same holds true even for square A, B. Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{but} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The former of those two examples show that $A \cdot B = \mathbf{0}$ does not yield that $A = \mathbf{0} \ \vee \ B = \mathbf{0}$.

There is no cancellation rule of matrix multiplication: $A \cdot B = A \cdot C$ and $A \neq \mathbf{0}$ does not necessarily yield $B = C$. Example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{but} \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Complexity of the multiplication of $(m \times n)$ by $(n \times k)$ matrices = the number of individual additions or multiplication. In the standard algorithm for each entry $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$ there are n multiplications and $n-1$ additions. Total: mnk multiplications and $m(n-1)k$ additions, is $O(n^3)$ for $(n \times n)$ matrices. There are much faster matrix multiplication algorithms than the standard one given by the definition above.

Example: Assume that we need to multiply three matrices A (10×10), B (10×100) and C (100×1). Since the matrix multiplication is associative, $A \cdot (B \cdot C) = (A \cdot B) \cdot C$, and the final result does not really depend on what multiplication is performed first. However, the complexity does! If we perform $A \cdot B$ first, we first spend $10 \cdot 10 \cdot 100$ number multiplications, and then $10 \cdot 100 \cdot 1$ number multiplications, which brings the total to $10000 + 1000 = 11000$. If $B \cdot C$ is taken first, then the total number of multiplications is $10 \cdot 100 \cdot 1 + 10 \cdot 10 \cdot 1 = 1100$ which is $1/10$ of the previous estimate.

Definition 11.4. Transpose of $[a_{ij}]^T = [a_{ji}]$. Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

A is **symmetric** if $A^T = A$ (note that such an A should be $(n \times n)$).

Definition 11.5. Identity matrix $I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 1 \end{pmatrix}$. $A \cdot I_n = I_n \cdot A = A$ holds for any $n \times n$ matrix A. The **power** of a square matrix A is defined by $A^0 = I_n$, $A^k = \underbrace{A \cdot A \cdot \ldots \cdot A}_{k \ times}$.

Definition 11.6 For 0-1 matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ we can define
$A \wedge B = [a_{ij} \wedge b_{ij}]$ (**meet**), $A \vee B = [a_{ij} \vee b_{ij}]$ (**join**),
$A \odot B = [(a_{i1} \wedge b_{1j}) \vee \ldots \vee (a_{in} \wedge b_{nj})]$ (**boolean product**),
$A^{[0]} = I_n$, $A^{[r+1]} = A^{[r]} \odot A$ (**boolean power**).

Homework assignments. (due Friday 02/23).
11A:Rosen2.6-2b; 11B:Rosen2.6-4c; 11C:Rosen2.6-24a; 11D:Rosen2.6-28