Exercise 1 (10 points) A sequence X_n is defined recursively by

$X_0 = a$

$X_1 = b$, where a, b are reals

$X_n = X_{n-1} + X_{n-2}$, for $n > 1$.

Prove by induction, that $X_n = b \cdot f_n + a \cdot f_{n-1}$ for all $n > 0$, where f_n is the nth Fibonacci number ($f_0 = 0$, $f_1 = 1$, $f_n = f_{n-2} + f_{n-1}$ for $n > 1$).

Solution: We shall prove by induction that $X_n = b \cdot f_n + a \cdot f_{n-1}$ for all $n > 0$ (*).

First of all, this is true for $n = 1$ since $X_1 = b = b \cdot 1 + a \cdot 0 = b \cdot f_1 + a \cdot f_0$. This is true for $n = 2$ also since $X_2 = X_1 + X_0 = b \cdot 1 + a \cdot 1 = b \cdot f_2 + a \cdot f_1$.

Given an arbitrary $n > 1$, let us assume (*) is true for $n - 1$ and for n.

$$X_{n+1} = X_n + X_{n-1}$$

Applying our induction hypothesis twice, we get

$$\begin{aligned} X_{n+1} &= \big(b \cdot f_n + a \cdot f_{n-1}\big) + \big(b \cdot f_{n-1} + a \cdot f_{n-2}\big) \\ &= b \cdot (f_n + f_{n-1}) + a \cdot (f_{n-1} + f_{n-2}) \\ &= b \cdot f_{n+1} + a \cdot f_n \end{aligned}$$

The last line comes from the Fibonacci property.

Please note that, in the inductive step when $n+1 = 3$, we used the fact that (*) holds for both $n = 2$ *and* $n = 1$. Therefore we needed to prove *two* consecutive base cases (the X_2 case as well as the X_1 case).

Exercise 2 (10 points) How many different arrangements can be formed using the letters *Manhattan*?

Solution: In *Manhattan* there are 3 a, 2 n, 2 t, 1 m and 1 h for a total of 9 letters. Theorem 22.9 from the notes gives the number of different arrangement via th following formula. If you don't recall it, you can still find it saying there are 9! arrangements if all the letters were different, and you count a same arrangement $n_i!$ times when letter i appears n_i times.

$$n = \frac{9!}{3! \cdot 2! \cdot 2! \cdot 1! \cdot 1! \cdot 1!} = 15120$$

Exercise 3 (10 points) How many positive integers with five digits or less have neither their first digit equal to 3 nor their last digit equal to 5?

Solution: We are going to count the relevant number of integers by considering successively integers with exactly one digit, then exactly 2 digits, etc up to 5 digits, each time with the formula: total number of integers minus those which are disqualified.

1. Between 0 and 9, there are $10 - 1 - 1 - 1 = 7$ relevant numbers. (0 is not positive,3 and 5 are not allowed).
2. Between 10 and 99, there are $90 - 10 - 9 + 1 = 72$ relevant numbers. There are 90 numbers between 10 and 99. Among them 10 begin by a 3 (30, 31, 32, ... , 39). Among them 9 end by a 5 (15, 25, 35, 45, etc.). And we counted twice 35, so we add 1. Another way to count is the following: 8 possibilities for the first digit (shouldn't be 0 nor 3), and 9 for the last one, that is $8 \times 9 = 72$
3. Between 100 and 999, there are $900 - 100 - 90 + 10 = 720$. Same reasoning, there are 900 numbers between 100 and 999, among them 100 start by a 3 and 90 end by a 5. If you don't see the 90, it comes from 100 (all number between 0 and 999 ending by a 5, minus the 10 ones which only have 1 or 2 digits. Here again we have counted 305, 315, 325, etc. twice.

   Or, 8 possibilities for the first digit (not 0 nor 3), 10 for the one in the middle, and 9 for the last one (not 5), $8 \times 10 \times 9 = 720$.
4. Between 1000 and 9999, there are $9,000 - 1000 - 900 + 100 = 8 \times 10 \times 10 \times 9 = 7,200$.
5. Between 10000 and 99999, there are $90,000 - 10,000 - 9,000 + 1,000 = 8 \times 10 \times 10 \times 10 \times 9 = 72,000$.

There are N of those numbers:

$$N = 7 + 72 + 720 + 7,200 + 72,000 = 79,999$$

Exercise 4 (10 points) Using Pascal's Triangle expand $(2x - y)^7$. Draw the corresponding portion of the Triangle. Feel free not to simplify the coefficients.

Solution:

$$\begin{aligned}(2x-y)^7 &= \sum_{i=0}^{7} C_7^i (-1)^{7-i} 2^i x^i y^{7-i} \\ &= 2^7x^7 - 7\cdot 2^6x^6y + 21\cdot 2^5x^5y^2 - 35\cdot 2^4x^4y^3 \\ &\quad + 35\cdot 2^3x^3y^4 - 21\cdot 2^2x^2y^5 + 7\cdot 2xy^6 - y^7 \\ &= 128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 \\ &\quad + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7\end{aligned}$$

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Pascal's Triangle expanded to level 7

Exercise 5 (10 points) How many positive integers are there less than 10000 such that the sum of their decimal digits is 12?

Solution: We have

$$x_1 + x_2 + x_3 + x_4 = 12 \tag{1}$$

Where the x_i's are nonnegative integer; as usual for this type of problem x_i represents the i^{th} digit.

The number we are looking for is the number of solutions to (1) subject to the constraint (C).

$$x_1 \leq 9 \text{ and } x_2 \leq 9 \text{ and } x_3 \leq 9 \text{ and } x_4 \leq 9 \tag{C}$$

The number of solutions to (1) subject to (C) is the number of solutions without constraints minus the number of solutions with the complement (D_1) of the constraint (C).

$$x_1 \geq 10 \text{ or } x_2 \geq 10 \text{ or } x_3 \geq 10 \text{ or } x_4 \geq 10 \tag{D_1}$$

If we look at the complement of the constraint: we get a disjunction as the constraint.

Now, the key observation is that the four disjuncts are pairwise disjoint, since if x_i and x_j ($i \neq j$) are both larger or equal to 10, then the sum clearly exceeds 12. So the number of solutions to (1) subject to (D_1) is simply four times the number of solutions to (1) subject to (D_2).

$$x_1 \geq 10 \tag{D_2}$$

The standard techniques ("stars and bars") yield:

(1) unconstrained:	12 + 4 - 1	choose 12	$C_{15}^{12} = 455$ solutions
(1) subject to (D_2):	2 + 4 - 1	choose 2	$C_5^2 = 10$ solutions

Subtracting, we get the number of solutions of (1) subject to (C): $455 - 4 \times 10 = 415$ solutions.

Exercise 6 (10 points) The deck of cards contains 52 cards. There are 13 different kinds of cards: 2,3,4,5,6,7,8,9,10,J,D,K,A. There are also four suits: spades, clubs, hearts, and diamonds, each containing 13 cards, with one card of each kind in a suit. What is the probability that a given poker hand of five cards is a royal flush (A,K,D,J,10 of the same suit)?

Solution: There are only four ways of getting a royal flush: (A$\spadesuit$ K$\spadesuit$ D$\spadesuit$ J$\spadesuit$ 10$\spadesuit$), (A$\heartsuit$ K$\heartsuit$ D$\heartsuit$ J$\heartsuit$ 10$\heartsuit$), (A$\diamondsuit$ K$\diamondsuit$ D$\diamondsuit$ J$\diamondsuit$ 10$\diamondsuit$), (A$\clubsuit$ K$\clubsuit$ D$\clubsuit$ J$\clubsuit$ 10$\clubsuit$).

There is a total of C_{52}^5 possible hands. Therefore the probability of getting a royal flush is $P = \frac{4}{C_{52}^5} \approx 1.54\ 10^{-6}$.

Note: Especially in probabilities, it is very easy to get things wrong, you should *always* explain what you do, a bunch of numbers doesn't prove anything, and nobody has a clue of what you are doing. I gave more credits to wrong answers with explanations than to the presumably same wrong answers without explanations.

Exercise 7 (10 points) A fair coin is tossed five times. What is the probability of getting exactly four heads, given that at least one of the tosses is heads?

Solution:

The number of relevant possibilities is $C_5^4 = C_5^1 = 5$, and the total number of possibilities is all those which have at least one heads, that is 2^5 minus the number of possibilities with no heads at all, which is 1. Therefore $P = \frac{5}{2^5 - 1} = \frac{5}{31} \approx 0.16$.

Exercise 8 (20 points) Some tribe values boys so much that each of their families keeps making kids until they get a boy (after which they relax and make no more kids). On the other hand, no family can afford having more than five kids. So if the first five babies in a family are girls, the family stops making children anyway. Assuming that a boy and a girl are equally likely, consider two random variables X - "the number of boys in a family", Y - "the number of girls in a family".

a) Find the expected values $E(X)$ and $E(Y)$ and compare them.
b) Are X and Y independent?
c) Find the expected value $E(X+Y)$.

Solution:

A family will always raise exactly one boy, except if they get 5 girls, in which case they won't get any boy. So $P(\text{one boy}) = 1 - \frac{1}{2^5} = \frac{31}{32}$ and $E(\text{number of boys}) = E(X) = \frac{31}{32} \approx 0.97$ boys.

$$
\begin{aligned}
P(\text{no girl}) = P(Y=0) &= \frac{1}{2} && \text{A boy as their first kid}\\
P(\text{one girl}) = P(Y=1) &= \frac{1}{2}\cdot\frac{1}{2} && \text{A girl and then a boy}\\
P(\text{two girls}) = P(Y=2) &= (\frac{1}{2})^3 && \text{A girl, a girl again, and then a boy}\\
P(\text{three girls}) = P(Y=3) &= (\frac{1}{2})^4 && \text{A girl three times, and then a boy}\\
P(\text{four girls}) = P(Y=4) &= (\frac{1}{2})^5 && \text{A girl four times, and then a boy}\\
P(\text{five girls}) = P(Y=5) &= (\frac{1}{2})^5 && \text{Five girls, and then they stop}
\end{aligned}
$$

$$
\begin{aligned}
E(\text{number of girls}) = E(Y) &= \sum_{i=0}^{5} i \cdot P(Y=i)\\
&= \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{32}\\
&= \frac{8+8+6+4+5}{32} = \frac{31}{32}\\
&\approx 0.97 \text{ girls}
\end{aligned}
$$

a) The expected number of boys is equal to the expected number of girls.
b) X and Y are certainly not independent since $(Y=5) \Rightarrow (X=0)$.
c) $E(X+Y)$ is always $E(X)+E(Y)$ even when X and Y are not independent.

$$E(X+Y) = 2 \cdot \frac{31}{32} = \frac{31}{16} \approx 1.94 \text{ children}$$

You can also consider Z the random variable number of children $Z = X+Y$, compute the $P(Z=i) = \frac{1}{2^i} = P(Y=i-1)$ for $i \in \{1,2,3,4\}, P(Z=5) = \frac{1}{2^4} = P(Y=4) = P(Y=5)$. And then $E(Z) = \sum_{i=1}^{5} i \cdot P(Z=i) = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{16} = \frac{8+8+6+4+5}{16} = \frac{31}{16}$, but this is a waste of time.