Exercise 5 (10 points) How many positive integers are there less than 10000 such that the sum of their decimal digits is 12?

Solution: We have

$$x_1 + x_2 + x_3 + x_4 = 12 \qquad (1)$$

Where the x_i's are nonnegative integer; as usual for this type of problem x_i represents the i^{th} digit.

The number we are looking for is the number of solutions to (1) subject to the constraint (C).

$$x_1 \leq 9 \text{ and } x_2 \leq 9 \text{ and } x_3 \leq 9 \text{ and } x_4 \leq 9 \qquad \text{(C)}$$

The number of solutions to (1) subject to (C) is the number of solutions without constraints minus the number of solutions with the complement (D_1) of the constraint (C).

$$x_1 \geq 10 \text{ or } x_2 \geq 10 \text{ or } x_3 \geq 10 \text{ or } x_4 \geq 10 \qquad (D_1)$$

If we look at the complement of the constraint: we get a disjunction as the constraint.

Now, the key observation is that the four disjuncts are pairwise disjoint, since if x_i and x_j $(i \neq j)$ are both larger or equal to 10, then the sum clearly exceeds 12. So the number of solutions to (1) subject to (D_1) is simply four times the number of solutions to (1) subject to (D_2).

$$x_1 \geq 10 \qquad (D_2)$$

The standard techniques (”stars and bars”) yield:

(1) unconstrained: 12 + 4 - 1 choose 12 $C_{15}^{12} = 455$ solutions
(1) subject to (D_2): 2 + 4 - 1 choose 2 $C_5^2 = 10$ solutions

Subtracting, we get the number of solutions of (1) subject to (C): $455 - 4 \times 10 = 415$ solutions.