**CS 280, Prelim 2 solutions November 9, 1999**

1. a) BASIS STEP: $n = 1$, $7 - 2 = 5$ is divisible by 5. INDUCTIVE STEP. Induction Hypothesis: $7^k - 2^k$ is divisible by 5.

$$7^{k+1}-2^{k+1} = 7^k \cdot 7 - 2^k \cdot 2 = 7^k \cdot 7 - 2^k \cdot 7 + 2^k \cdot 7 - 2^k \cdot 2 = 7(7^k - 2^k) + 2^k(7-2) = 7(7^k - 2^k) + 2^k \cdot 5$$

is divisible by 5 since $7^k - 2^k$ is.

b) The "INDUCTIVE STEP" fails when $k = 2$. Indeed, then $X = \{a, b\}$, $X - \{a\} = \{b\}$, $X - \{b\} = \{a\}$ and there is no way to conclude that $a = b$.

2. a) By the product rule, the number of ways to combine a team is $C(2, 1) \cdot C(18, 10)$.
b) By the binomial theorem, the coefficient of $x^4y^7$ is $-C(11, 4) \cdot 3^4 \cdot 2^7$.
c) By the formula of permutations with indistinguishable objects (12 letters, four Es, two Ss), the number is

$$\frac{12!}{4! \cdot 2!}.$$

d) First, the number of nonnegative integer solutions of the inequality $x + y \leq 19$ with $x \leq 15$ and $y \leq 15$ is equal to the number of nonnegative integer solutions of the equation $x+y+z = 19$ with $x \leq 15$ and $y \leq 15$. The total number of solutions of $x + y + z = 19$ can be evaluated by the formula of combinations with repetitions: $C(3+19-1, 19) = C(21, 19) = C(21, 2) = 210$. Let $A$ be the set of all solutions with $x \geq 16$, and $B$ be the set of all solutions with $y \geq 16$ respectively. By combinations with repetitions, $|A| = C(3+3-1, 3) = C(5, 3) = C(5, 2) = 10$. Similarly, $|B| = 10$. Note that $A \cap B = \emptyset$, since there are no solutions with $x \geq 16$ and $y \geq 16$. By the inclusion-exclusion formula $|A \cup B| = |A| + |B| - |A \cap B| = |A| + |B| = 20$, and the total number of solutions of the equation $x + y + z = 19$ with $x \leq 15$ and $y \leq 15$ is 210-20=190.

3. a) By the product rule, there are $B = C(52, 13) \cdot C(39, 13) \cdot C(26, 13)$ different ways to deal bridge hands to four players.

b) If player number one gets all four aces, then the total number of ways to deal the rest of the deck is $A = C(48, 9) \cdot C(39, 13) \cdot C(26, 13)$. The number of hands when some player gets all aces is then $4A$, and the probability of such an event is $4A/B$.

c) Use the formula for conditional probability: $p(E|F) = p(E \cap F)/p(F)$. Here $E$ is the set of hands where one player has all the aces, $F$ is the set of hands where player number one does not have aces. Note, that by the product rule, $|F| = C(48, 13) \cdot C(39, 13) \cdot C(26, 13)$ and $p(F) = |F|/B$. $E \cap F$ is the set of hands where all four aces go to one of the players number two, three or four. From the above, $|E \cap F| = 3A$ and $p(E \cap F) = 3A/B$. Therefore, $p(E|F) = p(E \cap F)/p(F) = 3A/|F|$.

4. a) $m = 1$ is supported by 11 variants out of 36, therefore $p(m = 1) = 11/36$. Similarly, $p(m = 2) = 9/36$, $p(m = 3) = 7/36$, $p(m = 4) = 5/36$, $p(m = 5) = 3/36$ and $p(m = 6) = 1/36$.

The expected value of $m$ is thus 11/36+18/36+21/36+20/36+15/36+6/36=91/36. Likewise, the expected value of $M$ is 1/36+6/36+15/36+28/36+45/36+66/36=161/36.

b) Random variables $m$ and $M$ are not independent. Indeed, $p(m = 6) = p(M = 1) = 1/36$, whereas $p(m = 6 \text{ and } M = 1) = 0 \neq p(m = 6) \cdot p(M = 1)$.

c) $E(m+M) = E(m)+E(M) = 91/36+161/36 = 252/36 = 7$. Another solution: for any possible outcome of the dice $m+M$ is nothing but the sum of the numbers $X_1+X_2$ that appear when the dice are rolled. Since $E(X_1) = E(X_2) = 7/2$, $E(X_1 + X_2) = E(X_1) + E(X_2) = 7$.

d) Since $m$ and $M$ are not independent, we cannot use the formula $E(m{\cdot}M) = E(m){\cdot}E(M)$ (this formula is just wrong in this case). Use the above argument instead. The product $m \cdot M$ for each outcome is nothing but the product $X_1 \cdot X_2$ Therefore $E(m \cdot M) = E(X_1 \cdot X_2)$. Unlike $m$ and $M$ the random variables $X_1$ and $X_2$ are independent, and $E(X_1{\cdot}X_2) = E(X_1){\cdot}E(X_2) = 7/2 \cdot 7/2 = 49/4$